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Global solutions to the 3-D incompressible inhomogeneous Navier–Stokes system

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Abstract

In this paper, we consider the global well-posedness of the 3-D incompressible inhomogeneous Navier–Stokes equations with initial data in the critical Besov spaces $a_0 \in B^{\frac{3}{q}}_{q,1}(\mathbb{R}^3)$, $u_0 = (u_0^h, u_0^3) \in B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)$ for p, q satisfying $1 < q \leqslant p < 6$ and $\frac{1}{q} - \frac{1}{p} \leqslant \frac{1}{3}$. More precisely, we prove that there exist two positive constants c_0, C_0 such that if $(\mu\|a_0\|_{B^{\frac{3}{q}}_{q,1}} + \|u_0^h\|_{B^{-1+\frac{3}{p}}_{p,1}}) \exp(C_0\|u_0^3\|^2_{B^{-1+\frac{3}{p}}_{p,1}}/\mu^2) \leqslant c_0\mu$, then (1.3) has a unique global solution $a \in \widetilde{L}^\infty(\mathbb{R}^+; B^{\frac{3}{q}}_{q,1}(\mathbb{R}^3))$, $u \in \widetilde{L}^\infty(\mathbb{R}^+; B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)) \cap L^1(\mathbb{R}^+; B^{1+\frac{3}{p}}_{p,1}(\mathbb{R}^3))$. In particular, this result implies the global well-posedness result in Abidi and Paicu (2007) [2] for the inhomogeneous Navier–Stokes system with small initial data.

Keywords: Inhomogeneous Navier–Stokes equations; Littlewood–Paley theory; Well-posedness

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1. Introduction

In this paper, we consider the global well-posedness of the following 3-D incompressible inhomogeneous Navier–Stokes equations with initial data in the critical Besov spaces

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(\mu \mathcal{M}) + \nabla \Pi = 0, \\ \operatorname{div} u = 0, \\ \rho|_{t=0} = \rho_0, \quad \rho u|_{t=0} = m_0, \end{cases} \tag{1.1}$$

where ρ, $u = (u_1, u_2, u_3)$ stand for the density and velocity of the fluid respectively, $\mathcal{M} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$, Π is a scalar pressure function, and in general, the viscosity coefficient $\mu(\rho)$ is a smooth, positive function on $[0, \infty)$. Such system describes a fluid which is obtained by mixing two miscible fluids that are incompressible and that have different densities. It may also describe a fluid containing a melted substance. One may check [18] for the detailed derivation of this system.

When $\mu(\rho)$ is independent of ρ, i.e. μ is a positive constant, and ρ_0 is bounded away from 0, Kazhikov [16] proved that: the inhomogeneous Navier–Stokes equations (1.1) have at least one global weak solutions in the energy space. In addition, he also proved the global existence of strong solutions to this system for small data in three space dimensions and all data in two dimensions. However, the uniqueness of both type weak solutions has not be solved. Ladyženskaja and Solonnikov [17] first addressed the question of unique resolvability of (1.1). More precisely, they considered the system (1.1) in bounded domain Ω with homogeneous Dirichlet boundary condition for u. Under the assumption that $u_0 \in W^{2-\frac{2}{p},p}(\Omega)$ $(p > N)$ is divergence free and vanishes on $\partial\Omega$ and that $\rho_0 \in C^1(\Omega)$ is bounded away from zero, then they [17] proved

- global well-posedness in dimension $N = 2$;
- local well-posedness in dimension $N = 3$. If in addition u_0 is small in $W^{2-\frac{2}{p},p}(\Omega)$, then global well-posedness holds true.

Similar results were obtained by Danchin [11] in $\mathbb{R}^N$ with initial data in the almost critical Sobolev spaces.

In the general case when $\mu(\rho)$ depends on ρ, DiPerna and Lions [13,18] proved the global existence of weak solutions to (1.1) in any space dimensions. Yet the uniqueness and regularities of such weak solutions are big open questions even in two space dimension, as was mentioned by Lions in [18]. On the other hand, Abidi, Gui and Zhang [3] investigated the large time decay and stability to any given global smooth solutions of (1.1), which in particular implies the global well-posedness of 3-D inhomogeneous Navier–Stokes equations with axi-symmetric initial data and without swirl for the initial velocity field provided that the initial density is close enough to a positive constant.

When the density ρ is away from zero, we denote by $a \stackrel{\text{def}}{=} \frac{1}{\rho} - 1$ and $\tilde{\mu}(a) \stackrel{\text{def}}{=} \mu(\rho)$, then the system (1.1) can be equivalently reformulated as

$$\text{(INS)} \quad \begin{cases} \partial_t a + u \cdot \nabla a = 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^N, \\ \partial_t u + u \cdot \nabla u + (1+a)\big(\nabla \Pi - \operatorname{div}(\tilde{\mu}(a)\mathcal{M})\big) = 0, \\ \operatorname{div} u = 0, \\ (a,u)|_{t=0} = (a_0, u_0). \end{cases}$$

Notice that just as the classical Navier–Stokes system, the inhomogeneous Navier–Stokes system (INS) also has a scaling. More precisely, if (a,u) solves (INS) with initial data (a_0,u_0), then for $\forall \ell > 0$,

$$(a,u)_\ell \stackrel{\text{def}}{=} \big(a\big(\ell^2\cdot,\ell\cdot\big), \ell u\big(\ell^2\cdot,\ell\cdot\big)\big) \quad \text{and} \quad (a_0,u_0)_\ell \stackrel{\text{def}}{=} \big(a_0(\ell\cdot), \ell u_0(\ell\cdot)\big) \tag{1.2}$$

$(a,u)_\ell$ is also a solution of (INS) with initial data $(a_0,u_0)_\ell$.

In [10], Danchin studied in general space dimension N the unique solvability of the system (INS) with constant viscosity coefficient and in scaling invariant (or critical) homogeneous Besov spaces, which generalized the celebrated results by Fujita and Kato [14] devoted to the classical Navier–Stokes system. In particular, the norm of $(a,u) \in B^{\frac{N}{2}}_{2,\infty}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \times B^{\frac{N}{2}-1}_{2,1}(\mathbb{R}^N)$ is scaling invariant under the change of scale of (1.2). In this case, Danchin proved that if the initial data $(a_0,u_0) \in B^{\frac{N}{2}}_{2,\infty}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \times B^{\frac{N}{2}-1}_{2,1}(\mathbb{R}^N)$ with a_0 sufficiently small in $B^{\frac{N}{2}}_{2,\infty}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, then the system (INS) has a unique local-in-time solution. In [1], Abidi proved that if $1 < p < 2N$, $0 < \underline{\mu} < \tilde{\mu}(a)$, $u_0 \in B^{\frac{N}{p}-1}_{p,1}(\mathbb{R}^N)$ and $a_0 \in B^{\frac{N}{p}}_{p,1}(\mathbb{R}^N)$, then (INS) has a global solution provided that $\|a_0\|_{B^{\frac{N}{p}}_{p,1}} + \|u_0\|_{B^{\frac{N}{p}-1}_{p,1}} \leqslant c_0$ for some c_0 sufficiently small. Furthermore, thus obtained solution is unique if $1 < p \leqslant N$. This result generalized the corresponding results in [10,11] and was improved by Abidi and Paicu in [2] (see Theorem 1.1 below) when $\tilde{\mu}(a)$ is a positive constant.

For simplicity, in what follows we just take $\mu(\rho) = \mu$ and the space dimension $N = 3$. In this case, (INS) becomes

$$\begin{cases} \partial_t a + u \cdot \nabla a = 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ \partial_t u + u \cdot \nabla u + (1+a)(\nabla \Pi - \mu \Delta u) = 0, \\ \operatorname{div} u = 0, \\ (a,u)|_{t=0} = (a_0,u_0). \end{cases} \tag{1.3}$$

Before we present our main result in this paper, let us recall the following result from [2]:

Theorem 1.1. *(See [2].) Let q, p satisfy $q,p \in (1,\infty)$ so that $\sup(\frac{1}{p},\frac{1}{q}) \leqslant \frac{1}{3} + \inf(\frac{1}{p},\frac{1}{q})$ and $\frac{1}{p} + \frac{1}{q} > \frac{1}{3}$. Let $a_0 \in B^{\frac{3}{q}}_{q,1}(\mathbb{R}^3)$, $u_0 \in B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)$ with $\|a_0\|_{B^{\frac{3}{q}}_{q,1}} \leqslant c$ for some sufficiently small c. Then (1.3) has a local solution (a,u) on $[0,T]$ such that*

$$\begin{aligned} &a \in \mathcal{C}\big([0,T]; B^{\frac{3}{q}}_{q,1}\big(\mathbb{R}^3\big)\big) \cap \widetilde{L}^\infty_T\big(B^{\frac{3}{q}}_{q,1}\big(\mathbb{R}^3\big)\big), \quad \nabla \Pi \in L^1_T\big(B^{-1+\frac{3}{p}}_{p,1}\big(\mathbb{R}^3\big)\big), \\ &u \in \mathcal{C}\big([0,T]; B^{-1+\frac{3}{p}}_{p,1}\big(\mathbb{R}^3\big)\big) \cap \widetilde{L}^\infty_T\big(B^{-1+\frac{3}{p}}_{p,1}\big(\mathbb{R}^3\big)\big) \cap L^1_T\big(B^{1+\frac{3}{p}}_{p,1}\big(\mathbb{R}^3\big)\big). \end{aligned}$$

Moreover, if $\|u_0\|_{B^{-1+\frac{3}{p}}_{p,1}} \leqslant c'\mu$ for c' small enough, then $T = \infty$. If $\frac{1}{p} + \frac{1}{q} \geqslant \frac{2}{3}$, then the solution is unique.

Motivated by [19,23,15] concerning the global well-posedness of 3-D incompressible anisotropic Navier–Stokes system with the third component of the initial velocity field being large, we are going to relax the smallness condition in Theorem 1.1 so that (1.3) still has a unique global solution. We emphasize that our proof uses in a fundamental way the algebraical structure of (1.3). The first step is to obtain energy estimates on the horizontal components of the velocity field on the one hand and then on the vertical component on the other hand. Compared with [19,23,15], the additional difficulties with this strategy are that: there appears a hyperbolic type equation in (1.3) and due to the appearance of a in the momentum equation of (1.3), the pressure term is more difficult to be estimated. We remark that the equation on the vertical component of the velocity is a linear equation with coefficients depending on the horizontal components and a. Therefore, the equation on the vertical component does not demand any smallness condition. While the equation on the horizontal component contain bilinear terms in the horizontal components and also terms taking into account the interactions between the horizontal components and the vertical one. In order to solve this equation, we need a smallness condition on a and the horizontal component (amplified by the vertical component) of the initial data. The main mathematical tool we shall use here will be a weighted Chemin–Lerner type norm introduced in [19].

For the convenience of readers, we recall the following form of weighted Chemin–Lerner type norm from [19]:

Definition 1.1. Let $f(t) \in L^1_{loc}(\mathbb{R}_+)$, $f(t) \geqslant 0$. We define

$$\|u\|_{\widetilde{L}^q_{T,f}(B^s_{p,r})} = \left\{ \sum_{\ell \in \mathbb{Z}} 2^{r\ell s} \left(\int_0^T f(t) \|\Delta_\ell u(t)\|^q_{L^p}\, dt \right)^{\frac{r}{q}} \right\}^{\frac{1}{r}}$$

for $s \in \mathbb{Z}$, $p \in [1, \infty]$, $q, r \in [1, \infty)$, and with the standard modification for $q = \infty$ or $r = \infty$.

The object of this paper is to prove the following theorem.

Theorem 1.2. *Let $1 < q \leqslant p < 6$ with $\frac{1}{q} - \frac{1}{p} \leqslant \frac{1}{3}$. There exist positive constants c_0 and C_0 such that, for any data $a_0 \in B^{\frac{3}{q}}_{q,1}(\mathbb{R}^3)$ and $u_0 = (u_0^h, u_0^3) \in B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)$ verifying*

$$\eta \overset{\text{def}}{=} \big(\mu \|a_0\|_{B^{\frac{3}{q}}_{q,1}} + \|u_0^h\|_{B^{-1+\frac{3}{p}}_{p,1}}\big) \exp\big\{C_0 \|u_0^3\|^2_{B^{-1+\frac{3}{p}}_{p,1}} / \mu^2\big\} \leqslant c_0 \mu, \tag{1.4}$$

(1.3) has a unique global solution $a \in \mathcal{C}([0,\infty); B^{\frac{3}{q}}_{q,1}(\mathbb{R}^3)) \cap \widetilde{L}^\infty(\mathbb{R}^+; B^{\frac{3}{q}}_{q,1}(\mathbb{R}^3))$ and $u \in \mathcal{C}([0,\infty); B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)) \cap \widetilde{L}^\infty(\mathbb{R}^+; B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)) \cap L^1(\mathbb{R}^+, B^{1+\frac{3}{p}}_{p,1}(\mathbb{R}^3))$. Moreover, there holds

$$\begin{aligned} &\|u - e^{\mu t \Delta} u_0\|_{\widetilde{L}^\infty(\mathbb{R}^+; B^{-1+\frac{3}{p}}_{p,1})} + \mu \|u - e^{\mu t \Delta} u_0\|_{L^1(\mathbb{R}^+; B^{1+\frac{3}{p}}_{p,1})} \\ &\quad \leqslant C\big(\|u_0^3\|_{B^{-1+\frac{3}{p}}_{p,1}} + \mu + \eta\big) \frac{\eta}{\mu}. \end{aligned} \tag{1.5}$$

Remark 1.1. (a) We remark that the restriction on p, q with $\sup(\frac{1}{p}, \frac{1}{q}) \leqslant \frac{1}{3} + \inf(\frac{1}{p}, \frac{1}{q})$ in Theorem 1.1 leads to $\frac{1}{q} - \frac{1}{p} \leqslant \frac{1}{3}$ in the case when $q \leqslant p$. However, the condition that $1 < q \leqslant p < 6$ is more restrictive than $\frac{1}{p} + \frac{1}{q} > \frac{1}{3}$ in Theorem 1.1. The main technical reason lies in the fact that here we need to deal with term like $u^3 \operatorname{div}_h u^h$ in the process to the estimate of the pressure term, which requires $p < 6$ (see the proof of Proposition 4.1 for details).

(b) We assert that our theorem remains to be true in the case when the viscosity coefficient depends on the density by a regular function $\mu(\rho)$ with $\mu(\rho) \geqslant \mu > 0$. In this case, we just need a small modification of the proof to Theorem 1.2 by using the fact that: for any positive s, we have $\|\tilde{\mu}(a) - \tilde{\mu}(0)\|_{B^s_{q,1}} \leqslant C(1 + \|a\|_{L^\infty})^{[s]+1} \|a\|_{B^s_{q,1}}$, where $\tilde{\mu}(a) = \mu(\frac{1}{1+a})$. We can also have a version of Theorem 1.2 in any space dimension. Just for a clear presentation, we choose to work in three space dimension here.

(c) Very recently Danchin and Mucha [12] can relax the smoothness condition for the density function, which in particular allows special discontinuous function, through Lagrangian approach. We should point out that here we crucially use the divergence condition of the velocity field. Yet in the Lagrangian coordinate, we still do not know how to use this condition to prove a similar version Theorem 1.2 in the framework of [12].

Remark 1.2. We emphasize that for any given function a, ϕ in the Schwartz space $\mathcal{S}(\mathbb{R}^3)$, p, q satisfying $3 < q < p < 6$ and $\frac{1}{q} - \frac{1}{p} < \frac{1}{3}$, Theorem 1.2 implies the global well-posedness of (1.3) with initial data of the form

$$
\begin{aligned}
a_0^\varepsilon(x) &= \sqrt{-\delta \ln \varepsilon}\, \varepsilon^{(1+\frac{2}{q})\alpha} a\big(x_1, x_2/\varepsilon^\alpha, x_3\big) \quad \text{and} \\
u_0^\epsilon &= \frac{\mu}{\sqrt{C}} \sqrt{-\delta \ln \varepsilon}\, \varepsilon^{-(1-\frac{3}{p}+\frac{\alpha}{p})} \sin\Big(\frac{x_1}{\varepsilon}\Big)\big(0, -\varepsilon^\alpha \partial_3 \phi, \partial_2 \phi\big)\big(x_1, x_2/\varepsilon^\alpha, x_3\big),
\end{aligned}
$$

for $\alpha \in (0, 3(1 - \frac{1}{p}))$, $0 < \delta < \alpha$, ε sufficiently small and C large enough.

Indeed one gets by applying Lemma 3.1 of [7] that

$$
\Big\| \sin\Big(\frac{x_1}{\varepsilon}\Big) \nabla \phi\big(x_1, x_2/\varepsilon^\alpha, x_3\big) \Big\|_{B_{p,1}^{-1+\frac{3}{p}}} \leqslant C_\phi \varepsilon^{1-\frac{3}{p}+\frac{\alpha}{p}}. \tag{1.6}
$$

Whereas thanks to Definition 2.1 below, we have

$$
\begin{aligned}
\big\|a\big(x_1, x_2/\varepsilon^\alpha, x_3\big)\big\|_{B_{q,1}^{\frac{3}{q}}} &= \sum_{j<N} 2^{\frac{3j}{q}} \big\|\Delta_j\big[a\big(x_1, x_2/\varepsilon^\alpha, x_3\big)\big]\big\|_{L^q} \\
&\quad + \sum_{j \geqslant N} 2^{\frac{3j}{q}} \big\|\Delta_j\big[a\big(x_1, x_2/\varepsilon^\alpha, x_3\big)\big]\big\|_{L^q} \\
&\leqslant C\bigg(\sum_{j<N} \varepsilon^{\frac{\alpha}{q}} 2^{\frac{3j}{q}} \|a\|_{L^q} + \sum_{j \geqslant N} 2^{-j(1-\frac{3}{q})} \varepsilon^{\alpha(\frac{1}{q}-1)} \|\nabla a\|_{L^q} \bigg) \\
&\leqslant C\big(\varepsilon^{\frac{\alpha}{q}} 2^{\frac{3N}{q}} \|a\|_{L^q} + 2^{-N(1-\frac{3}{q})} \varepsilon^{\alpha(\frac{1}{q}-1)} \|\nabla a\|_{L^q}\big),
\end{aligned}
$$

choosing the best integer N in the above inequality leads to

$$\left\|a\left(x_1, x_2/\varepsilon^\alpha, x_3\right)\right\|_{B^{\frac{3}{q}}_{q,1}} \leqslant C\varepsilon^{-\frac{2\alpha}{q}}\|a\|_{L^q}^{1-\frac{3}{q}}\|\nabla a\|_{L^q}^{\frac{3}{q}},$$

which along with (1.6) shows that

$$\left(\mu\left\|a_0^\varepsilon\right\|_{B^{\frac{3}{q}}_{q,1}} + \left\|u_0^{\varepsilon,h}\right\|_{B^{-1+\frac{3}{p}}_{p,1}}\right)\exp\left\{C_0\left\|u_0^{\varepsilon,3}\right\|^2_{B^{-1+\frac{3}{p}}_{p,1}}/\mu^2\right\} \leqslant C\sqrt{-\delta\ln\varepsilon}\,\varepsilon^{\alpha-\delta}\mu,$$

which implies that $(a_0^\varepsilon, u_0^\varepsilon)$ satisfies (1.4) for ε sufficiently small. Then Theorem 1.2 ensures that (1.3) with initial data $(a_0^\varepsilon, u_0^\varepsilon)$ has a unique global solution $(a^\varepsilon, u^\varepsilon)$ such that

$$\begin{aligned}&\left\|u^\varepsilon - e^{\mu t\Delta}u_0^\varepsilon\right\|_{\widetilde{L}^\infty(\mathbb{R}^+; B^{-1+\frac{3}{p}}_{p,1})} + \mu\left\|u^\varepsilon - e^{\mu t\Delta}u_0^\varepsilon\right\|_{L^1(\mathbb{R}^+; B^{1+\frac{3}{p}}_{p,1})}\\ &\quad\leqslant C\mu\sqrt{-\delta\ln\varepsilon}\left(1+\sqrt{-\delta\ln\varepsilon}\right)\varepsilon^{\alpha-\delta} \to 0 \quad \text{as } \varepsilon \to 0.\end{aligned}$$

We should point out that compared with the smallness condition in [2], here the norms of our data $(a_0^\varepsilon, u_0^\varepsilon)$ have been amplified by $\sqrt{-\delta\ln\varepsilon}$.

Remark 1.3. In the case of the classical Navier–Stokes system,

$$(\mathrm{NS})\qquad \begin{cases}\partial_t u + u\cdot\nabla u - \mu\Delta u = -\nabla\Pi,\\ \operatorname{div} u = 0,\\ \left(u^h, u^3\right)\big|_{t=0} = \left(u_0^h, u_0^3\right),\end{cases}$$

which corresponds to $a = 0$ in (1.3). A small modification to the proof of Theorem 1.2 ensures that: for any $1 < p < 6$, $1 \leqslant r < \infty$, given $u_0 \in B^{-1+\frac{3}{p}}_{p,r}(\mathbb{R}^3)$ with $\operatorname{div} u_0 = 0$ and

$$\bar\eta \stackrel{\text{def}}{=} \left\|u_0^h\right\|_{B^{-1+\frac{3}{p}}_{p,r}}\exp\left\{C\left\|u_0^3\right\|^{2r}_{B^{-1+\frac{3}{p}}_{p,r}}/\mu^{2r}\right\} \leqslant c\mu \tag{1.7}$$

for some positive constants c, C. Then (NS) has a unique solutions $u \in \mathcal{C}([0,\infty); B^{-1+\frac{3}{p}}_{p,r}(\mathbb{R}^3)) \cap \widetilde{L}^\infty(\mathbb{R}^+; B^{-1+\frac{3}{p}}_{p,r}(\mathbb{R}^3)) \cap \widetilde{L}^1(\mathbb{R}^+; B^{1+\frac{3}{p}}_{p,r}(\mathbb{R}^3))$. Moreover, there holds

$$\left\|u - e^{\mu t\Delta}u_0\right\|_{\widetilde{L}^\infty(\mathbb{R}^+; B^{-1+\frac{3}{p}}_{p,r})} + \mu\left\|u - e^{\mu t\Delta}u_0\right\|_{\widetilde{L}^1(\mathbb{R}^+; B^{1+\frac{3}{p}}_{p,r})} \leqslant C\left(\left\|u_0^3\right\|_{B^{-1+\frac{3}{p}}_{p,1}} + \mu + \bar\eta\right)\frac{\bar\eta}{\mu}.$$

The main idea of the proof will be the application of the weighted Chemin–Lerner type norm, $\widetilde{L}^1_{t,f}(B^{-1+\frac{3}{p}}_{p,r}(\mathbb{R}^3))$, for $f(t) = \|u^3(t)\|^{2r}_{B^{-1+\frac{3}{p}+\frac{1}{r}}_{p,r}}$. Then one can obtain this result for (NS) by following exactly the same line as the proof of Theorem 1.2.

We should mention that the main reason that we can relax the smallness condition in (1.4) for (1.3) to (1.7) for (NS) is that: to propagating the regularities for a in (1.3), we need the convection velocity $u \in L^1([0,T]; Lip(\mathbb{R}^3))$. Due to this additional difficulty of the transport equation in (1.3), here we choose to present the detailed proof of Theorem 1.2 for the inhomogeneous Navier–Stokes system.

Let us complete this section by the organization of the paper:

Scheme of the proof and organization of the paper In the second section, we shall collect some basic facts on Littlewood–Paley analysis; then in Section 3 we apply the Littlewood–Paley theory to study the transport equation in the framework of weighted Chemin–Lerner type norms. In Section 4, we shall present the estimate to the pressure function. Finally in the last section, we shall complete the proof of Theorem 1.2.

2. Preliminaries

2.1. Notation

Let A, B be two operators, we denote $[A;B]=AB-BA$, the commutator between A and B. For $a\lesssim b$, we mean that there is a uniform constant C, which may be different on different lines, such that $a\leqslant Cb$. We shall denote by $(a\mid b)$ the $L^2(\mathbb{R}^3)$ inner product of a and b. $(d_j)_{j\in\mathbb{Z}}$ will be a generic element of $\ell^1(\mathbb{Z})$ so that $d_j\geqslant 0$ and $\sum_{j\in\mathbb{Z}}d_j=1$.

For X a Banach space and I an interval of $\mathbb{R}$, we denote by $\mathcal{C}(I;X)$ the set of continuous functions on I with values in X, and by $L^q(I;X)$ stands for the set of measurable functions on I with values in X, such that $t\mapsto\|f(t)\|_X$ belongs to $L^q(I)$.

2.2. Littlewood–Paley theory

The proof of Theorem 1.2 requires a dyadic decomposition of the Fourier variables, or Littlewood–Paley decomposition. Let us briefly explain how it may be built in the case $x\in\mathbb{R}^3$ (see e.g. [4]). Let φ be a smooth function supported in the ring $\mathcal{C}\stackrel{\text{def}}{=}\{\xi\in\mathbb{R}^3,\ \frac{3}{4}\leqslant|\xi|\leqslant\frac{8}{3}\}$ and such that

$$\sum_{j\in\mathbb{Z}}\varphi\big(2^{-j}\xi\big)=1\quad\text{for }\xi\neq 0.$$

For $u\in\mathcal{S}'(\mathbb{R}^3)$, we set

$$\forall j\in\mathbb{Z},\quad \Delta_j u\stackrel{\text{def}}{=}\varphi\big(2^{-j}\mathrm{D}\big)u\quad\text{and}\quad S_j u\stackrel{\text{def}}{=}\sum_{\ell\leqslant j-1}\Delta_\ell u.$$

Then we have the formal decomposition

$$u=\sum_{j\in\mathbb{Z}}\Delta_j u,\quad \forall u\in\mathcal{S}'\big(\mathbb{R}^3\big)/\mathcal{P}\big[\mathbb{R}^3\big],$$

where $\mathcal{P}[\mathbb{R}^3]$ is the set of polynomials (see [20]). Moreover, the Littlewood–Paley decomposition satisfies the property of almost orthogonality:

$$\Delta_k\Delta_j u\equiv 0\quad\text{if }|k-j|\geqslant 2\quad\text{and}\quad\Delta_k(S_{j-1}u\Delta_j u)\equiv 0\quad\text{if }|k-j|\geqslant 5. \tag{2.1}$$

We recall now the definition of homogeneous Besov spaces from [22].

Definition 2.1. Let $(p,r)\in[1,+\infty]^2$, $s\in\mathbb{R}$ and $u\in\mathcal{S}'(\mathbb{R}^3)$, we set

$$\|u\|_{B^s_{p,r}}\stackrel{\mathrm{def}}{=}\big(2^{qs}\|\Delta_q u\|_{L^p}\big)_{\ell^r}.$$

- For $s<\frac{3}{p}$ (or $s=\frac{3}{p}$ if $r=1$), we define $B^s_{p,r}(\mathbb{R}^3)\stackrel{\mathrm{def}}{=}\{u\in\mathcal{S}'(\mathbb{R}^3)\mid \|u\|_{B^s_{p,r}}<\infty\}$.
- If $k\in\mathbb{N}$ and $\frac{3}{p}+k\leqslant s<\frac{3}{p}+k+1$ (or $s=\frac{3}{p}+k+1$ if $r=1$), then $B^s_{p,r}(\mathbb{R}^3)$ is defined as the subset of distributions $u\in\mathcal{S}'(\mathbb{R}^3)$ such that $\partial^\beta u\in B^{s-k}_{p,r}(\mathbb{R}^3)$ whenever $|\beta|=k$.

Remark 2.1.

(1) It is easy to observe that the homogeneous Besov space $B^s_{2,2}(\mathbb{R}^3)$ coincides with the classical homogeneous Sobolev space $\dot H^s(\mathbb{R}^3)$ and $B^s_{\infty,\infty}(\mathbb{R}^3)$ coincides with the classical homogeneous Hölder space $\dot C^s(\mathbb{R}^3)$ when s is not a positive integer, in the case when s is a nonnegative integer, $B^s_{\infty,\infty}(\mathbb{R}^3)$ coincides with the classical homogeneous Zygmund space $\dot C^s_*(\mathbb{R}^3)$.

(2) Let $s\in\mathbb{R}$, $1\leqslant p,r\leqslant\infty$, and $u\in\mathcal{S}'(\mathbb{R}^3)$. Then u belongs to $B^s_{p,r}(\mathbb{R}^3)$ if and only if there exists $\{c_{j,r}\}_{j\in\mathbb{Z}}$ such that $c_{j,r}\geqslant 0$, $\|c_{j,r}\|_{\ell^r}=1$ and

$$\|\Delta_j u\|_{L^p}\leqslant Cc_{j,r}2^{-js}\|u\|_{B^s_{p,r}}\quad\text{for all } j\in\mathbb{Z}.$$

In order to obtain a better description of the regularizing effect of the transport-diffusion equation, we will use Chemin–Lerner type spaces $\widetilde{L}^\lambda_T(B^s_{p,r}(\mathbb{R}^3))$ from [8,6].

Definition 2.2. Let $s\leqslant\frac{3}{p}$ (respectively $s\in\mathbb{R}$), $(r,\lambda,p)\in[1,+\infty]^3$ and $T\in\,]0,+\infty]$. We define $\widetilde{L}^\lambda_T(B^s_{p,r}(\mathbb{R}^3))$ as the completion of $C([0,T];\mathcal{S}(\mathbb{R}^3))$ by the norm

$$\|f\|_{\widetilde{L}^\lambda_T(B^s_{p,r})}\stackrel{\mathrm{def}}{=}\Bigg(\sum_{q\in\mathbb{Z}}2^{qrs}\bigg(\int_0^T\big\|\Delta_q f(t)\big\|^\lambda_{L^p}\,dt\bigg)^{\frac{r}{\lambda}}\Bigg)^{\frac{1}{r}}<\infty,$$

with the usual change if $r=\infty$. For short, we just denote this space by $\widetilde{L}^\lambda_T(B^s_{p,r})$.

For the convenience of the reader, we recall some basic facts on Littlewood–Paley theory, one may check [4,5,22] for more details.

Lemma 2.1. *Let $\mathcal{B}$ be a ball and $\mathcal{C}$ a ring of $\mathbb{R}^3$. A constant C exists so that for any positive real number λ, any nonnegative integer k, any homogeneous function σ of degree m smooth outside of 0 and any couple of real numbers (a,b) with $b\geqslant a\geqslant 1$, there hold*

$$\begin{aligned}
&\operatorname{Supp}\hat u\subset\lambda\mathcal{B}\quad\Rightarrow\quad \sup_{|\alpha|=k}\big\|\partial^\alpha u\big\|_{L^b}\leqslant C^{k+1}\lambda^{k+3(\frac{1}{a}-\frac{1}{b})}\|u\|_{L^a},\\
&\operatorname{Supp}\hat u\subset\lambda\mathcal{C}\quad\Rightarrow\quad C^{-1-k}\lambda^k\|u\|_{L^a}\leqslant\sup_{|\alpha|=k}\big\|\partial^\alpha u\big\|_{L^a}\leqslant C^{1+k}\lambda^k\|u\|_{L^a},\\
&\operatorname{Supp}\hat u\subset\lambda\mathcal{C}\quad\Rightarrow\quad \big\|\sigma(D)u\big\|_{L^b}\leqslant C_{\sigma,m}\lambda^{m+3(\frac{1}{a}-\frac{1}{b})}\|u\|_{L^a}.
\end{aligned}\tag{2.2}$$

In the sequel, we shall frequently use Bony's decomposition from [5] in the homogeneous context:

$$uv = T_u v + T_v u + R(u, v), \tag{2.3}$$

where

$$T_u v \stackrel{\text{def}}{=} \sum_{j\in\mathbb{Z}} S_{j-1}u\Delta_j v, \qquad R(u,v) \stackrel{\text{def}}{=} \sum_{j\in\mathbb{Z}} \Delta_j u \widetilde{\Delta}_j v, \quad \text{and} \quad \widetilde{\Delta}_j v \stackrel{\text{def}}{=} \sum_{|j'-j|\leqslant 1} \Delta_{j'} v.$$

As an application of the above basic facts on Littlewood–Paley theory, we present the following product laws in Besov spaces, which will be constantly used in the sequel. One may check [2] for more general product laws in this respect.

Lemma 2.2. *Let $p_2 \geqslant p_1 \geqslant 1$, and $s_1 \leqslant \frac{3}{p_1}$, $s_2 \leqslant \frac{3}{p_2}$ with $s_1 + s_2 > 3\max(0, \frac{1}{p_1} + \frac{1}{p_2} - 1)$. Let $a \in B^{s_1}_{p_1,1}(\mathbb{R}^3)$, $b \in B^{s_2}_{p_2,1}(\mathbb{R}^3)$. Then $ab \in B^{s_1+s_2-\frac{3}{p_1}}_{p_2,1}(\mathbb{R}^3)$, and*

$$\|ab\|_{B^{s_1+s_2-\frac{3}{p_1}}_{p_2,1}} \lesssim \|a\|_{B^{s_1}_{p_1,1}} \|b\|_{B^{s_2}_{p_2,1}}.$$

Proof. The proof of this lemma is standard, for completeness, we outline its proof here. First thanks to (2.3), we have

$$ab = T_a b + T_b a + R(a, b).$$

Applying Lemma 2.1 gives

$$\big\|\Delta_j(T_a b)\big\|_{L^{p_2}} \lesssim \sum_{|j'-j|\leqslant 5} \|S_{j'-1}a\|_{L^\infty} \|\Delta_{j'} b\|_{L^{p_2}} \lesssim d_j 2^{j(\frac{3}{p_1}-s_1-s_2)} \|a\|_{B^{s_1}_{p_1,1}} \|b\|_{B^{s_2}_{p_2,1}},$$

and

$$\begin{aligned}\big\|\Delta_j(T_b a)\big\|_{L^{p_2}} &\lesssim \sum_{|j'-j|\leqslant 5} 2^{3j'(\frac{1}{p_1}-\frac{1}{p_2})} \|S_{j'-1}b\|_{L^\infty} \|\Delta_{j'} a\|_{L^{p_1}} \\ &\lesssim d_j 2^{j(\frac{3}{p_1}-s_1-s_2)} \|a\|_{B^{s_1}_{p_1,1}} \|b\|_{B^{s_2}_{p_2,1}},\end{aligned}$$

as $s_1 \leqslant \frac{3}{p_1}$, $\|S_{j'-1}a\|_{L^\infty} \lesssim 2^{j(\frac{3}{p_1}-s_1)} \|a\|_{B^{s_1}_{p_1,1}}$, and a similar estimate for $\|S_{j'-1}b\|_{L^\infty}$. Finally, in the case where $\frac{1}{p_1} + \frac{1}{p_2} \leqslant 1$, notice that $s_1 + s_2 > 0$, we get by using Lemma 2.1 once again that

$$\big\|\Delta_j\big(R(a,b)\big)\big\|_{L^{p_2}} \lesssim 2^{\frac{3j}{p_1}} \sum_{j'\geqslant j-N_0} \|\Delta_{j'} a\|_{L^{p_1}} \|\widetilde{\Delta}_{j'} b\|_{L^{p_2}}$$

$$\begin{aligned}
&\lesssim 2^{\frac{3j}{p_1}} \sum_{j' \geqslant j-N_0} d_{j'} 2^{-j'(s_1+s_2)} \|a\|_{B^{s_1}_{p_1,1}} \|b\|_{B^{s_2}_{p_2,1}} \\
&\lesssim d_j 2^{j(\frac{3}{p_1}-s_1-s_2)} \|a\|_{B^{s_1}_{p_1,1}} \|b\|_{B^{s_2}_{p_2,1}}.
\end{aligned}$$

If $\frac{1}{p_1}+\frac{1}{p_2}>1=\frac{1}{\lambda}+\frac{1}{p_2}$, we use the following estimates

$$\begin{aligned}
\big\|\Delta_j\big(R(a,b)\big)\big\|_{L^{p_2}} &\lesssim 2^{3j(1-\frac{1}{p_2})} \sum_{j' \geqslant j-N_0} \|\Delta_{j'} a \widetilde{\Delta}_{j'} b\|_{L^1} \\
&\lesssim 2^{3j(1-\frac{1}{p_2})} \sum_{j' \geqslant j-N_0} \|\Delta_{j'} a\|_{L^\lambda} \|\widetilde{\Delta}_{j'} b\|_{L^{p_2}} \\
&\lesssim 2^{3j(1-\frac{1}{p_2})} \sum_{j' \geqslant j-N_0} 2^{3j'(\frac{1}{p_1}-\frac{1}{\lambda})} \|\Delta_{j'} a\|_{L^{p_1}} \|\widetilde{\Delta}_{j'} b\|_{L^{p_2}} \\
&\lesssim 2^{3j(1-\frac{1}{p_2})} \sum_{j' \geqslant j-N_0} 2^{3j'(\frac{1}{p_1}+\frac{1}{p_2}-1)} d_{j'} 2^{-j'(s_1+s_2)} \|a\|_{B^{s_1}_{p_1,1}} \|b\|_{B^{s_2}_{p_2,1}} \\
&\lesssim d_j 2^{j(\frac{3}{p_1}-s_1-s_2)} \|a\|_{B^{s_1}_{p_1,1}} \|b\|_{B^{s_2}_{p_2,1}},
\end{aligned}$$

for $s_1+s_2>3(\frac{1}{p_1}+\frac{1}{p_2}-1)$. This completes the proof of the lemma. □

3. The estimate of the transport equation

The goal of this section is to investigate the following free transport equation in the framework of weighted Chemin–Lerner type norms:

$$\partial_t a + u\cdot\nabla a = 0, \qquad a|_{t=0}=a_0. \tag{3.1}$$

In particular, we shall prove the following proposition:

Proposition 3.1. *Let $p \geqslant q > 1$ with $\frac{1}{q}-\frac{1}{p}\leqslant\frac{1}{3}$ and λ be a positive number. Let $u=(u^h,u^3)\in\widetilde{L}^\infty_T(B^{-1+\frac{3}{p}}_{p,1})\cap L^1_T(B^{1+\frac{3}{p}}_{p,1})$ and $a_0\in B^{\frac{3}{q}}_{q,1}(\mathbb{R}^3)$. We denote $f(t)\stackrel{\text{def}}{=}\|u^3(t)\|_{B^{1+\frac{3}{p}}_{p,1}}$ and $a_\lambda\stackrel{\text{def}}{=}a\exp(-\lambda\int_0^t f(t')\,dt')$. Then* (3.1) *has a unique solution $a\in\mathcal{C}([0,T];B^{\frac{3}{q}}_{q,1}(\mathbb{R}^3))$ so that*

$$\|a_\lambda\|_{\widetilde{L}^\infty_t(B^{\frac{3}{q}}_{q,1})} + \frac{\lambda}{2}\|a_\lambda\|_{\widetilde{L}^1_{t,f}(B^{\frac{3}{q}}_{q,1})} \leqslant \|a_0\|_{B^{\frac{3}{q}}_{q,1}} + C\big\|u^h\big\|_{L^1_t(B^{1+\frac{3}{p}}_{p,1})}\|a_\lambda\|_{\widetilde{L}^\infty_t(B^{\frac{3}{q}}_{q,1})} \tag{3.2}$$

for any $t\in[0,T]$ and λ large enough.

Remark 3.1. In fact, we will only use the estimate (3.2) in the case of $\operatorname{div} u=0$, which is not used in the proof of Proposition 3.1.

Proof of Proposition 3.1. The existence and uniqueness of solutions to (3.1) essentially follow from the estimate (3.2) for some appropriate approximate solutions to (3.1). For simplicity, here we just present the estimate (3.2) for smooth enough solutions of (3.1). In this case, thanks to (3.1), we have

$$\partial_t a_\lambda + \lambda f(t) a_\lambda + u \cdot \nabla a_\lambda = 0.$$

Applying Δ_j to the above equation and taking L^2 inner product of the resulting equation with $|\Delta_j a_\lambda|^{q-2}\Delta_j a_\lambda$ (when $q \in (1,2)$, we need to make some modification as that in [9]), we obtain

$$\frac{1}{q}\frac{d}{dt}\left\|\Delta_j a_\lambda(t)\right\|_{L^q}^q + \lambda f(t)\left\|\Delta_j a_\lambda(t)\right\|_{L^q}^q + \left(\Delta_j(u\cdot\nabla a_\lambda) \,\middle|\, |\Delta_j a_\lambda|^{q-2}\Delta_j a_\lambda\right) = 0. \tag{3.3}$$

On the other hand, we get by using Bony's decomposition (2.3) that

$$u\cdot\nabla a_\lambda = T_u \nabla a_\lambda + T_{\nabla a_\lambda} u + R(u, \nabla a_\lambda),$$

and one get by using a standard commutator's argument that

$$\begin{aligned}
\left(\Delta_j(T_u\nabla a_\lambda) \,\middle|\, |\Delta_j a_\lambda|^{q-2}\Delta_j a_\lambda\right) = {} & \sum_{|j'-j|\leqslant 5}\left([\Delta_j; S_{j'-1}u]\Delta_{j'}\nabla a_\lambda \,\middle|\, |\Delta_j a_\lambda|^{q-2}\Delta_j a_\lambda\right)\\
& + \sum_{|j'-j|\leqslant 5}\left((S_{j'-1}u - S_{j-1}u)\Delta_j\Delta_{j'}\nabla a_\lambda \,\middle|\, |\Delta_j a_\lambda|^{q-2}\Delta_j a_\lambda\right)\\
& - \frac{1}{q}\left(S_{j-1}(\operatorname{div} u)\Delta_j a_\lambda \,\middle|\, |\Delta_j a_\lambda|^{q-2}\Delta_j a_\lambda\right).
\end{aligned}$$

Then thanks to (3.3) and using an argument for the L^q energy estimate in [9], we arrive at

$$\begin{aligned}
&\left\|\Delta_j a_\lambda(t)\right\|_{L^q} + \lambda\int_0^t f(t')\left\|\Delta_j a_\lambda(t')\right\|_{L^q}\,dt'\\
&\quad\leqslant \|\Delta_j a_0\|_{L^q}\\
&\qquad + C\Bigg\{\sum_{|j'-j|\leqslant 5}\left(\left\|[\Delta_j; S_{j'-1}u]\Delta_{j'}\nabla a_\lambda\right\|_{L^1_t(L^q)} + \left\|(S_{j'-1}u - S_{j-1}u)\Delta_j\Delta_{j'}\nabla a_\lambda\right\|_{L^1_t(L^q)}\right)\\
&\qquad + \left\|S_{j-1}(\operatorname{div} u)\Delta_j a_\lambda\right\|_{L^1_t(L^q)} + \|T_{\nabla a_\lambda}u\|_{L^1_t(L^q)} + \left\|R(u,\nabla a_\lambda)\right\|_{L^1_t(L^q)}\Bigg\}.
\end{aligned} \tag{3.4}$$

We first get by applying the classical estimate on commutators (see [4] for instance) that

$$\begin{aligned}
&\sum_{|j'-j|\leqslant 5}\left\|[\Delta_j; S_{j'-1}u]\Delta_{j'}\nabla a_\lambda\right\|_{L^1_t(L^q)}\\
&\quad\lesssim \sum_{|j'-j|\leqslant 5}\left(\left\|S_{j'-1}\nabla u^h\right\|_{L^1_t(L^\infty)}\|\Delta_{j'}a_\lambda\|_{L^\infty_t(L^q)} + \int_0^t\left\|S_{j'-1}\nabla u^3(t')\right\|_{L^\infty}\left\|\Delta_{j'}a_\lambda(t')\right\|_{L^q}\,dt'\right)
\end{aligned}$$

$$\lesssim \sum_{|j'-j|\leqslant 5}\Bigg(d_{j'}2^{-\frac{3j'}{q}}\|u^h\|_{L^1_t(B^{1+\frac{3}{p}}_{p,1})}\|a_\lambda\|_{\widetilde{L}^\infty_t(B^{\frac{3}{q}}_{q,1})}+\int_0^t\|u^3(t')\|_{B^{1+\frac{3}{p}}_{p,1}}\|\Delta_{j'}a_\lambda(t')\|_{L^q}\,dt'\Bigg)$$
$$\lesssim d_j 2^{-\frac{3j}{q}}\big(\|u^h\|_{L^1_t(B^{1+\frac{3}{p}}_{p,1})}\|a_\lambda\|_{\widetilde{L}^\infty_t(B^{\frac{3}{q}}_{q,1})}+\|a_\lambda\|_{\widetilde{L}^1_{t,f}(B^{\frac{3}{q}}_{q,1})}\big),$$

where we used Definition 1.1 in the last step. Similarly we get by applying Lemma 2.1 and Definition 1.1 that

$$\begin{aligned}
&\sum_{|j'-j|\leqslant 5}\big\|(S_{j'-1}u-S_{j-1}u)\Delta_j\Delta_{j'}\nabla a_\lambda\big\|_{L^1_t(L^q)}\\
&\quad\lesssim \sum_{|j'-j|\leqslant 5}\Bigg(\big\|\big(S_{j'-1}\nabla u^h-S_{j-1}\nabla u^h\big)\big\|_{L^1_t(L^\infty)}\|\Delta_j a_\lambda\|_{L^\infty_t(L^q)}\\
&\qquad+\int_0^t\big\|\big(S_{j'-1}\nabla u^3-S_{j-1}\nabla u^3\big)(t')\big\|_{L^\infty}\big\|\Delta_j a_\lambda(t')\big\|_{L^q}\,dt'\Bigg)\\
&\quad\lesssim d_j2^{-\frac{3j}{q}}\|u^h\|_{L^1_t(B^{1+\frac{3}{p}}_{p,1})}\|a_\lambda\|_{\widetilde{L}^\infty_t(B^{\frac{3}{q}}_{q,1})}+\sum_{|j'-j|\leqslant 5}\int_0^t\|u^3(t')\|_{B^{1+\frac{3}{p}}_{p,1}}\big\|\Delta_j a_\lambda(t')\big\|_{L^q}\,dt'\\
&\quad\lesssim d_j2^{-\frac{3j}{q}}\big(\|u^h\|_{L^1_t(B^{1+\frac{3}{p}}_{p,1})}\|a_\lambda\|_{\widetilde{L}^\infty_t(B^{\frac{3}{q}}_{q,1})}+\|a_\lambda\|_{\widetilde{L}^1_{t,f}(B^{\frac{3}{q}}_{q,1})}\big),
\end{aligned}$$

and

$$\begin{aligned}
\big\|S_{j-1}(\operatorname{div}u)\Delta_j a_\lambda\big\|_{L^1_t(L^q)}&\lesssim\big\|S_{j-1}\big(\operatorname{div}_h u^h\big)\big\|_{L^1_t(L^\infty)}\|\Delta_j a_\lambda\|_{L^\infty_t(L^q)}\\
&\quad+\int_0^t\big\|S_{j-1}\big(\partial_3 u^3\big)(t')\big\|_{L^\infty}\big\|\Delta_j a_\lambda(t')\big\|_{L^q}\,dt'\\
&\lesssim d_j2^{-\frac{3j}{q}}\big(\|u^h\|_{L^1_t(B^{1+\frac{3}{p}}_{p,1})}\|a_\lambda\|_{\widetilde{L}^\infty_t(B^{\frac{3}{q}}_{q,1})}+\|a_\lambda\|_{\widetilde{L}^1_{t,f}(B^{\frac{3}{q}}_{q,1})}\big).
\end{aligned}$$

On the other hand, as $q\leqslant p$, let r be determined by $\frac{1}{r}=\frac{1}{q}-\frac{1}{p}$. Then applying Lemma 2.1 ensures that

$$\begin{aligned}
\big\|\Delta_j(T_{\nabla a_\lambda}u)\big\|_{L^1_t(L^q)}&\lesssim\sum_{|j'-j|\leqslant 5}\Bigg(\|S_{j'-1}\nabla_h a_\lambda\|_{L^\infty_t(L^r)}\big\|\Delta_{j'}u^h\big\|_{L^1_t(L^p)}\\
&\quad+\int_0^t\big\|S_{j'-1}\partial_3 a_\lambda(t')\big\|_{L^r}\big\|\Delta_{j'}u^3(t')\big\|_{L^p}\,dt'\Bigg),
\end{aligned}\tag{3.5}$$

now as $\frac{1}{q}-\frac{1}{p}\leqslant\frac{1}{3}$, one has

$$\begin{aligned}\|S_{j'-1}\nabla_h a_\lambda\|_{L^\infty_t(L^r)} &\lesssim \sum_{\ell\leqslant j'-2} 2^{\ell(1+3(\frac{1}{q}-\frac{1}{r}))}\|\Delta_\ell a_\lambda\|_{L^\infty_t(L^q)}\\ &\lesssim \sum_{\ell\leqslant j'-2} d_\ell 2^{\ell(1+3(\frac{1}{p}-\frac{1}{q}))}\|a_\lambda\|_{\widetilde{L}^\infty_t(B^{\frac{3}{q}}_{q,1})}\lesssim 2^{j'(1+3(\frac{1}{p}-\frac{1}{q}))}\|a_\lambda\|_{\widetilde{L}^\infty_t(B^{\frac{3}{q}}_{q,1})},\end{aligned}$$

and applying Lemma 2.1 once again gives

$$\begin{aligned}&\sum_{|j'-j|\leqslant 5}\int_0^t \big\|S_{j'-1}\partial_3 a_\lambda(t')\big\|_{L^r}\big\|\Delta_{j'}u^3(t')\big\|_{L^p}\,dt'\\ &\lesssim 2^{-j(1+\frac{3}{p})}\sum_{|j'-j|\leqslant 5}\sum_{\ell\leqslant j'-2}2^{\ell(1+\frac{3}{p})}\int_0^t\big\|\Delta_\ell a_\lambda(t')\big\|_{L^q}\big\|u^3(t')\big\|_{B^{1+\frac{3}{p}}_{p,1}}\,dt'\\ &\lesssim 2^{-j(1+\frac{3}{p})}\sum_{\ell\leqslant j+3}d_\ell 2^{\ell(1+\frac{3}{p}-\frac{3}{q})}\|a_\lambda\|_{\widetilde{L}^1_{t,f}(B^{\frac{3}{q}}_{q,1})}\\ &\lesssim d_j 2^{-\frac{3j}{q}}\|a_\lambda\|_{\widetilde{L}^1_{t,f}(B^{\frac{3}{q}}_{q,1})}\quad \text{if } \frac{1}{q}-\frac{1}{p}<\frac{1}{3}.\end{aligned}$$

In the case when $\frac{1}{q}-\frac{1}{p}=\frac{1}{3}$, we have $1+\frac{3}{p}=\frac{3}{q}$, and

$$\begin{aligned}&\sum_{j\in\mathbb{Z}}2^{\frac{3j}{q}}\sum_{|j'-j|\leqslant 5}\int_0^t\big\|S_{j'-1}\partial_3 a_\lambda(t')\big\|_{L^r}\big\|\Delta_{j'}u^3(t')\big\|_{L^p}\,dt'\\ &\lesssim\sum_{j\in\mathbb{Z}}\sum_{\ell\leqslant j-2}2^{\ell(1+\frac{3}{p})}\int_0^t d_j(t')\big\|\Delta_\ell a_\lambda(t')\big\|_{L^q}\big\|u^3(t')\big\|_{B^{1+\frac{3}{p}}_{p,1}}\,dt'\\ &\lesssim\sum_\ell d_\ell\|a_\lambda\|_{\widetilde{L}^1_{t,f}(B^{\frac{3}{q}}_{q,1})}\lesssim\|a_\lambda\|_{\widetilde{L}^1_{t,f}(B^{\frac{3}{q}}_{q,1})}.\end{aligned}$$

As a consequence, we deduce from (3.5) that

$$\big\|\Delta_j(T_{\nabla a_\lambda}u)\big\|_{L^1_t(L^q)}\lesssim d_j 2^{-\frac{3j}{q}}\Big(\big\|u^h\big\|_{L^1_t(B^{1+\frac{3}{p}}_{p,1})}\|a_\lambda\|_{\widetilde{L}^\infty_t(B^{\frac{3}{q}}_{q,1})}+\|a_\lambda\|_{\widetilde{L}^1_{t,f}(B^{\frac{3}{q}}_{q,1})}\Big).$$

Finally let us turn to the last term in (3.4). In fact for the part involving only u^h, in the case where $\frac{1}{p}+\frac{1}{q}\leqslant 1$, we get by applying Lemma 2.1 that

$$\big\|\Delta_j\big(R\big(u^h,\nabla_h a_\lambda\big)\big)\big\|_{L^1_t(L^q)}\lesssim 2^{\frac{3j}{p}}\sum_{j'\geqslant j-N_0}\big\|\Delta_{j'}u^h\big\|_{L^1_t(L^p)}\|\widetilde{\Delta}_{j'}\nabla_h a_\lambda\|_{L^\infty_t(L^q)}$$

$$\begin{aligned}
&\lesssim 2^{\frac{3j}{p}} \sum_{j' \geqslant j-N_0} d_{j'} 2^{-3j'(\frac{1}{p}+\frac{1}{q})} \|u^h\|_{L^1_t(B^{1+\frac{3}{p}}_{p,1})} \|a_\lambda\|_{\widetilde{L}^\infty_t(B^{\frac{3}{q}}_{q,1})} \\
&\lesssim d_j 2^{-\frac{3j}{q}} \|u^h\|_{L^1_t(B^{1+\frac{3}{p}}_{p,1})} \|a_\lambda\|_{\widetilde{L}^\infty_t(B^{\frac{3}{q}}_{q,1})},
\end{aligned}$$

in the case where $\frac{1}{p}+\frac{1}{q}>1=\frac{1}{r}+\frac{1}{q}$, we get by applying Lemma 2.1 that

$$\begin{aligned}
&\big\|\Delta_j\big(R\big(u^h,\nabla_h a_\lambda\big)\big)\big\|_{L^1_t(L^q)} \\
&\quad\lesssim 2^{3j(1-\frac{1}{q})} \sum_{j' \geqslant j-N_0} \big\|\Delta_{j'} u^h\big\|_{L^1_t(L^r)} \|\widetilde{\Delta}_{j'} \nabla_h a_\lambda\|_{L^\infty_t(L^q)} \\
&\quad\lesssim 2^{3j(1-\frac{1}{q})} \sum_{j' \geqslant j-N_0} 2^{3j'(\frac{1}{p}-\frac{1}{r})} \big\|\Delta_{j'} u^h\big\|_{L^1_t(L^p)} \|\widetilde{\Delta}_{j'} \nabla_h a_\lambda\|_{\widetilde{L}^\infty_t(L^q)} \\
&\quad\lesssim 2^{3j(1-\frac{1}{q})} \sum_{j' \geqslant j-N_0} 2^{3j'(\frac{1}{p}-\frac{1}{r})} d_{j'} 2^{-3j'(\frac{1}{p}+\frac{1}{q})} \big\|u^h\big\|_{L^1_t(B^{1+\frac{3}{p}}_{p,1})} \|a_\lambda\|_{\widetilde{L}^\infty_t(B^{\frac{3}{q}}_{q,1})} \\
&\quad\lesssim 2^{3j(1-\frac{1}{q})} \sum_{j' \geqslant j-N_0} 2^{-3j'} d_{j'} \big\|u^h\big\|_{L^1_t(B^{1+\frac{3}{p}}_{p,1})} \|a_\lambda\|_{\widetilde{L}^\infty_t(B^{\frac{3}{q}}_{q,1})} \\
&\quad\lesssim d_j 2^{-\frac{3j}{q}} \big\|u^h\big\|_{L^1_t(B^{1+\frac{3}{p}}_{p,1})} \|a_\lambda\|_{\widetilde{L}^\infty_t(B^{\frac{3}{q}}_{q,1})}.
\end{aligned}$$

For the part involving u^3, in the case $\frac{1}{p}+\frac{1}{q}\leqslant 1$, we get by using Definition 1.1 that

$$\begin{aligned}
\big\|\Delta_j\big(R\big(u^3,\partial_3 a_\lambda\big)\big)\big\|_{L^1_t(L^q)} &\lesssim 2^{\frac{3j}{p}} \sum_{j' \geqslant j-N_0} \int_0^t \big\|\widetilde{\Delta}_{j'} u^3(t')\big\|_{L^p} \big\|\Delta_{j'} \partial_3 a_\lambda(t')\big\|_{L^q}\,dt' \\
&\lesssim 2^{\frac{3j}{p}} \sum_{j' \geqslant j-N_0} 2^{-\frac{3j'}{p}} \int_0^t \big\|u^3(t')\big\|_{B^{1+\frac{3}{p}}_{p,1}} \big\|\Delta_{j'} a_\lambda(t')\big\|_{L^q}\,dt' \\
&\lesssim 2^{\frac{3j}{p}} \sum_{j' \geqslant j-N_0} d_{j'} 2^{-3j'(\frac{1}{p}+\frac{1}{q})} \|a_\lambda\|_{\widetilde{L}^1_{t,f}(B^{\frac{3}{q}}_{q,1})} \\
&\lesssim d_j 2^{-\frac{3j}{q}} \|a_\lambda\|_{\widetilde{L}^1_{t,f}(B^{\frac{3}{q}}_{q,1})},
\end{aligned}$$

whereas in the case $\frac{1}{p}+\frac{1}{q}>1=\frac{1}{r}+\frac{1}{q}$ we can write

$$\begin{aligned}
&\big\|\Delta_j\big(R\big(u^3,\partial_3 a_\lambda\big)\big)\big\|_{L^1_t(L^q)} \\
&\quad\lesssim 2^{3j(1-\frac{1}{q})} \sum_{j' \geqslant j-N_0} \int_0^t \big\|\widetilde{\Delta}_{j'} u^3(t')\big\|_{L^r} \big\|\Delta_{j'} \partial_3 a_\lambda(t')\big\|_{L^q}\,dt'
\end{aligned}$$

$$
\begin{aligned}
&\lesssim 2^{3j(1-\frac{1}{q})}\sum_{j'\geqslant j-N_0}2^{3j'(\frac{1}{p}-\frac{1}{r})}2^{-\frac{3j'}{p}}\int_0^t \big\|u^3(t')\big\|_{B_{p,1}^{1+\frac{3}{p}}}\big\|\Delta_{j'}a_\lambda(t')\big\|_{L^q}\,dt'\\
&\lesssim 2^{3j(1-\frac{1}{q})}\sum_{j'\geqslant j-N_0}d_{j'}2^{-3j'(\frac{1}{r}+\frac{1}{q})}\|a_\lambda\|_{\widetilde{L}^1_{t,f}(B_{q,1}^{\frac{3}{q}})}\\
&\lesssim d_j 2^{-\frac{3j}{q}}\|a_\lambda\|_{\widetilde{L}^1_{t,f}(B_{q,1}^{\frac{3}{q}})}.
\end{aligned}
$$

Substituting the above estimates into (3.4) and taking summation for $j\in\mathbb{Z}$, we arrive at

$$
\|a\|_{\widetilde{L}^\infty_t(B_{q,1}^{\frac{3}{q}})}+\lambda\|a_\lambda\|_{\widetilde{L}^1_{t,f}(B_{q,1}^{\frac{3}{q}})}\leqslant\|a_0\|_{B_{q,1}^{\frac{3}{q}}}+C\Big(\big\|u^h\big\|_{L^1_t(B_{p,1}^{1+\frac{3}{p}})}\|a\|_{\widetilde{L}^\infty_t(B_{q,1}^{\frac{3}{q}})}+\|a_\lambda\|_{\widetilde{L}^1_{t,f}(B_{q,1}^{\frac{3}{q}})}\Big).
$$

Taking $\lambda\geqslant 2C$ in the above inequality, we conclude the proof of (3.2). □

4. The estimate of the pressure

As is well known, the main difficulty in the study of the well-posedness of incompressible inhomogeneous Navier–Stokes equations is to the derive the estimate for the pressure term. The goal of this section is to provide the pressure estimates in the framework of weighted Chemin–Lerner type norms. We first get by taking div to the momentum equation of (1.3) that

$$
-\Delta\Pi=\operatorname{div}(a\nabla\Pi)+\operatorname{div}_h\operatorname{div}_h\big(u^h\otimes u^h\big)+2\partial_3\operatorname{div}_h\big(u^3u^h\big)+\partial_3^2\big(u^3\big)^2-\mu\operatorname{div}(a\Delta u),\tag{4.1}
$$

where, for a vector field $v=(v^h,v^3)$ we denote $\operatorname{div}_h v^h=\partial_1v^1+\partial_2v^2$.

The following proposition concerning the estimate of the pressure will be the main ingredient used in the estimate of u^h.

Proposition 4.1. *Let* $1\leqslant q\leqslant p<6$ *and* $a\in\widetilde{L}^\infty_T(B_{q,1}^{\frac{3}{q}})$, $u\in\widetilde{L}^\infty_T(B_{p,1}^{-1+\frac{3}{p}})\cap L^1_T(B_{p,1}^{1+\frac{3}{p}})$. *We denote*

$$
f_1(t)\stackrel{\text{def}}{=}\big\|u^3(t)\big\|_{B_{p,1}^{1+\frac{3}{p}}},\qquad f_2(t)\stackrel{\text{def}}{=}\big\|u^3(t)\big\|^2_{B_{p,1}^{\frac{3}{p}}}\quad\text{and}
$$

$$
\Pi_{\bar\lambda}\stackrel{\text{def}}{=}\Pi\exp\bigg(-\lambda_1\int_0^t f_1(t')\,dt'-\lambda_2\int_0^t f_2(t')\,dt'\bigg)\quad\text{for }\lambda_1,\lambda_2>0,\tag{4.2}
$$

and similar notations for $a_{\bar\lambda}$ *and* $u_{\bar\lambda}$. *Then* (4.1) *has a unique solution* $\nabla\Pi\in L^1_T(B_{p,1}^{-1+\frac{3}{p}})$ *which decays to zero when* $|x|\to\infty$ *so that for all* $t\in[0,T]$, *there holds*

$$
\|\nabla\Pi_{\bar\lambda}\|_{L^1_t(B_{p,1}^{-1+\frac{3}{p}})}\leqslant\frac{C}{1-C\|a\|_{L^\infty_t(B_{q,1}^{\frac{3}{q}})}}\Big\{\Big(\mu\|a\|_{L^\infty_t(B_{q,1}^{\frac{3}{q}})}+\big\|u^h\big\|_{L^\infty_t(B_{p,1}^{-1+\frac{3}{p}})}\Big)\big\|u^h_{\bar\lambda}\big\|_{L^1_t(B_{p,1}^{1+\frac{3}{p}})}
$$

$$+\|u^h_{\bar\lambda}\|^{\frac12}_{L^1_t(B^{1+\frac3p}_{p,1})}\|u^h_{\bar\lambda}\|^{\frac12}_{\widetilde L^1_{t,f_2}(B^{-1+\frac3p}_{p,1})}+\|u^h_{\bar\lambda}\|_{\widetilde L^1_{t,f_1}(B^{-1+\frac3p}_{p,1})}$$
$$+\mu\|a_{\bar\lambda}\|_{\widetilde L^1_{t,f_1}(B^{\frac3q}_{q,1})}\Big\} \tag{4.3}$$

provided that $C\|a\|_{L^\infty_T(B^{\frac3q}_{q,1})}\leqslant\frac12$.

The proof of this proposition will mainly be based on the following lemmas:

Lemma 4.1. *Under the assumptions of Proposition* 4.1 *and* f_1, f_2 *being given by* (4.2)*, one has*

$$\|\Delta_j(u^3u^h)\|_{L^1_t(L^p)}\lesssim d_j2^{-\frac{3j}{p}}\Big(\|u^h\|^{\frac12}_{L^1_t(B^{1+\frac3p}_{p,1})}\|u^h\|^{\frac12}_{\widetilde L^1_{t,f_2}(B^{-1+\frac3p}_{p,1})}+\|u^h\|_{\widetilde L^1_{t,f_1}(B^{-1+\frac3p}_{p,1})}\Big)\quad\text{and}$$

$$\|\Delta_j(u^3u^h)\|_{L^1_t(L^p)}\lesssim d_j2^{-\frac{3j}{p}}\Big(\|u^h\|_{\widetilde L^\infty_t(B^{-1+\frac3p}_{p,1})}\|u^3\|_{L^1_t(B^{1+\frac3p}_{p,1})}+\|u^h\|_{L^1_t(B^{1+\frac3p}_{p,1})}\|u^3\|_{\widetilde L^\infty_t(B^{-1+\frac3p}_{p,1})}\Big).$$

Proof. Indeed thanks to Bony's decomposition, we have

$$\Delta_j(u^3u^h)=\sum_{j'\geqslant j-N_0}\Delta_j\big(S_{j'}u^3\Delta_{j'}u^h+\Delta_{j'}u^3S_{j'+1}u^h\big). \tag{4.4}$$

Notice that on the one hand,

$$\begin{aligned}\Big\|\sum_{j'\geqslant j-N_0}\Delta_j\big(S_{j'}u^3\Delta_{j'}u^h\big)(t')\Big\|_{L^p}&\lesssim\sum_{j'\geqslant j-N_0}\|S_{j'}u^3(t')\|_{L^\infty}\|\Delta_{j'}u^h(t')\|_{L^p}\\&\lesssim\sum_{j'\geqslant j-N_0}\|u^3(t')\|_{B^{\frac3p}_{p,1}}\|\Delta_{j'}u^h(t')\|_{L^p},\end{aligned}$$

integrating the above inequality over $[0,t]$ leads to

$$\begin{aligned}&\Big\|\sum_{j'\geqslant j-N_0}\Delta_j\big(S_{j'}u^3\Delta_{j'}u^h\big)\Big\|_{L^1_t(L^p)}\\&\quad\lesssim\sum_{j'\geqslant j-N_0}\bigg\{\int_0^t\|u^3(t')\|^2_{B^{\frac3p}_{p,1}}\|\Delta_{j'}u^h(t')\|_{L^p}\,dt'\bigg\}^{\frac12}\|\Delta_{j'}u^h\|^{\frac12}_{L^1_t(L^p)}\\&\quad\lesssim\sum_{j'\geqslant j-N_0}d_{j'}2^{-\frac{3j'}{p}}\|u^h\|^{\frac12}_{\widetilde L^1_{t,f_2}(B^{-1+\frac3p}_{p,1})}\|u^h\|^{\frac12}_{L^1_t(B^{1+\frac3p}_{p,1})}\\&\quad\lesssim d_j2^{-\frac{3j}{p}}\|u^h\|^{\frac12}_{L^1_t(B^{1+\frac3p}_{p,1})}\|u^h\|^{\frac12}_{\widetilde L^1_{t,f_2}(B^{-1+\frac3p}_{p,1})}.\end{aligned}$$

On the other hand, again thanks to Lemma 2.1, we obtain

$$\begin{aligned}\Big\|\sum_{j'\geqslant j-N_0}\Delta_j\big(\Delta_{j'}u^3S_{j'+1}u^h\big)(t')\Big\|_{L^p}&\lesssim\sum_{j'\geqslant j-N_0}\big\|S_{j'+1}u^h(t')\big\|_{L^\infty}\big\|\Delta_{j'}u^3(t')\big\|_{L^p}\\&\lesssim\sum_{j'\geqslant j-N_0}2^{-j'(1+\frac{3}{p})}\big\|u^3(t')\big\|_{B^{1+\frac{3}{p}}_{p,1}}\big\|S_{j'+1}u^h(t')\big\|_{L^\infty},\end{aligned}$$

integrating the above inequality over $[0,t]$ results in

$$\begin{aligned}&\Big\|\sum_{j'\geqslant j-N_0}\Delta_j\big(\Delta_{j'}u^3S_{j'+1}u^h\big)\Big\|_{L^1_t(L^p)}\\&\quad\lesssim\sum_{j'\geqslant j-N_0}2^{-j'(1+\frac{3}{p})}\sum_{\ell\leqslant j'}2^{\frac{3\ell}{p}}\int_0^t\big\|u^3(t')\big\|_{B^{1+\frac{3}{p}}_{p,1}}\big\|\Delta_\ell u^h(t')\big\|_{L^p}\,dt'\\&\quad\lesssim\sum_{j'\geqslant j-N_0}d_{j'}2^{-\frac{3j'}{p}}\big\|u^h\big\|_{\widetilde{L}^1_{t,f_1}(B^{-1+\frac{3}{p}}_{p,1})}\lesssim d_j2^{-\frac{3j}{p}}\big\|u^h\big\|_{\widetilde{L}^1_{t,f_1}(B^{-1+\frac{3}{p}}_{p,1})}.\end{aligned}$$

This proves the first inequality of the lemma.

Whereas again thanks to (4.4), we get by applying Lemma 2.1 that

$$\begin{aligned}&\big\|\Delta_j\big(u^3u^h\big)\big\|_{L^1_t(L^p)}\\&\quad\lesssim\sum_{j'\geqslant j-N_0}\big(\big\|S_{j'}u^3\big\|_{L^\infty_t(L^\infty)}\big\|\Delta_{j'}u^h\big\|_{L^1_t(L^p)}+\big\|\Delta_{j'}u^3\big\|_{L^1_t(L^p)}\big\|S_{j'+1}u^h\big\|_{L^\infty_t(L^\infty)}\big)\\&\quad\lesssim\sum_{j'\geqslant j-N_0}d_{j'}2^{-\frac{3j'}{p}}\big(\big\|u^h\big\|_{L^1_t(B^{1+\frac{3}{p}}_{p,1})}\big\|u^3\big\|_{\widetilde{L}^\infty_t(B^{-1+\frac{3}{p}}_{p,1})}+\big\|u^h\big\|_{\widetilde{L}^\infty_t(B^{-1+\frac{3}{p}}_{p,1})}\big\|u^3\big\|_{L^1_t(B^{1+\frac{3}{p}}_{p,1})}\big)\\&\quad\lesssim d_j2^{-\frac{3j}{p}}\big(\big\|u^h\big\|_{L^1_t(B^{1+\frac{3}{p}}_{p,1})}\big\|u^3\big\|_{\widetilde{L}^\infty_t(B^{-1+\frac{3}{p}}_{p,1})}+\big\|u^h\big\|_{\widetilde{L}^\infty_t(B^{-1+\frac{3}{p}}_{p,1})}\big\|u^3\big\|_{L^1_t(B^{1+\frac{3}{p}}_{p,1})}\big).\end{aligned}$$

This completes the proof of the lemma. □

Lemma 4.2. *Let $p<6$ and f_1, f_2 be given by* (4.2)*. Then under the assumptions of Proposition* 4.1*, one has*

$$\big\|\Delta_j\big(u^3\operatorname{div}_h u^h\big)\big\|_{L^1_t(L^p)}\lesssim d_j2^{j(1-\frac{3}{p})}\big(\big\|u^h\big\|^{\frac12}_{L^1_t(B^{1+\frac{3}{p}}_{p,1})}\big\|u^h\big\|^{\frac12}_{\widetilde{L}^1_{t,f_2}(B^{-1+\frac{3}{p}}_{p,1})}+\big\|u^h\big\|_{\widetilde{L}^1_{t,f_1}(B^{-1+\frac{3}{p}}_{p,1})}\big),$$

and

$$\begin{aligned}\big\|\Delta_j\big(u^3\operatorname{div}_h u^h\big)\big\|_{L^1_t(L^p)}\lesssim d_j2^{j(1-\frac{3}{p})}\big(&\big\|u^h\big\|_{\widetilde{L}^\infty_t(B^{-1+\frac{3}{p}}_{p,1})}\big\|u^3\big\|_{L^1_t(B^{1+\frac{3}{p}}_{p,1})}\\&+\big\|u^h\big\|_{L^1_t(B^{1+\frac{3}{p}}_{p,1})}\big\|u^3\big\|_{\widetilde{L}^\infty_t(B^{-1+\frac{3}{p}}_{p,1})}\big).\end{aligned}$$

Proof. We first get by applying Bony's decomposition (2.3) that

$$u^3 \operatorname{div}_h u^h = T_{u^3} \operatorname{div}_h u^h + T_{\operatorname{div}_h u^h} u^3 + R\big(u^3, \operatorname{div}_h u^h\big). \tag{4.5}$$

Applying Lemma 2.1 gives

$$\begin{aligned}\big\|\Delta_j\big(T_{u^3} \operatorname{div}_h u^h\big)(t')\big\|_{L^p} &\lesssim \sum_{|j'-j|\leqslant 5} 2^{j'} \big\|S_{j'-1} u^3(t')\big\|_{L^\infty} \big\|\Delta_{j'} u^h(t')\big\|_{L^p} \\ &\lesssim \sum_{|j'-j|\leqslant 5} 2^{j'} \big\|u^3(t')\big\|_{B^{\frac{3}{p}}_{p,1}} \big\|\Delta_{j'} u^h(t')\big\|_{L^p},\end{aligned}$$

integrating the above inequality over $[0,t]$ gives rise to

$$\begin{aligned}\big\|\Delta_j\big(T_{u^3} \operatorname{div}_h u^h\big)\big\|_{L^1_t(L^p)} &\lesssim \sum_{|j'-j|\leqslant 5} 2^{j'} \left\{\int_0^t \big\|u^3(t')\big\|^2_{B^{\frac{3}{p}}_{p,1}} \big\|\Delta_{j'} u^h(t')\big\|_{L^p}\, dt'\right\}^{\frac{1}{2}} \big\|\Delta_j u^h\big\|^{\frac{1}{2}}_{L^1_t(L^p)} \\ &\lesssim d_j 2^{j(1-\frac{3}{p})} \big\|u^h\big\|^{\frac{1}{2}}_{L^1_t(B^{1+\frac{3}{p}}_{p,1})} \big\|u^h\big\|^{\frac{1}{2}}_{\widetilde{L}^1_{t,f_2}(B^{-1+\frac{3}{p}}_{p,1})}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\big\|\Delta_j\big(T_{\operatorname{div}_h u^h} u^3\big)(t')\big\|_{L^p} &\lesssim \sum_{|j'-j|\leqslant 5} \big\|S_{j'-1} \operatorname{div}_h u^h(t')\big\|_{L^\infty} \big\|\Delta_{j'} u^3(t')\big\|_{L^p} \\ &\lesssim \sum_{|j'-j|\leqslant 5} 2^{-j'(1+\frac{3}{p})} \big\|u^3(t')\big\|_{B^{1+\frac{3}{p}}_{p,1}} \big\|S_{j'-1} \operatorname{div}_h u^h(t')\big\|_{L^\infty} \\ &\lesssim 2^{-j(1+\frac{3}{p})} \sum_{\ell\leqslant j+4} 2^{\ell(1+\frac{3}{p})} \big\|u^3(t')\big\|_{B^{1+\frac{3}{p}}_{p,1}} \big\|\Delta_\ell u^h(t')\big\|_{L^p},\end{aligned}$$

from which, we deduce that

$$\begin{aligned}\big\|\Delta_j\big(T_{\operatorname{div}_h u^h} u^3\big)\big\|_{L^1_t(L^p)} &\lesssim 2^{-j(1+\frac{3}{p})} \sum_{\ell\leqslant j+4} d_\ell 2^{2\ell} \big\|u^h\big\|_{\widetilde{L}^1_{t,f_1}(B^{-1+\frac{3}{p}}_{p,1})} \\ &\lesssim d_j 2^{j(1-\frac{3}{p})} \big\|u^h\big\|_{\widetilde{L}^1_{t,f_1}(B^{-1+\frac{3}{p}}_{p,1})}.\end{aligned}$$

Finally, for $2 \leqslant p < 6$, we get by applying Lemma 2.1 that

$$\begin{aligned}\big\|\Delta_j\big(R\big(u^3, \operatorname{div}_h u^h\big)\big)\big\|_{L^1_t(L^p)} &\lesssim 2^{\frac{3j}{p}} \sum_{j'\geqslant j-N_0} 2^{j'} \int_0^t \big\|\widetilde{\Delta}_{j'} u^3(t')\big\|_{L^p} \big\|\Delta_{j'} u^h(t')\big\|_{L^p}\, dt' \\ &\lesssim 2^{\frac{3j}{p}} \sum_{j'\geqslant j-N_0} 2^{-\frac{3j'}{p}} \int_0^t \big\|u^3(t')\big\|_{B^{1+\frac{3}{p}}_{p,1}} \big\|\Delta_{j'} u^h(t')\big\|_{L^p}\, dt'\end{aligned}$$

$$
\begin{aligned}
&\lesssim 2^{\frac{3j}{p}} \sum_{j' \geqslant j-N_0} d_{j'} 2^{j'(1-\frac{6}{p})} \|u^h\|_{\widetilde{L}^1_{t,f_1}(B^{-1+\frac{3}{p}}_{p,1})} \\
&\lesssim d_j 2^{j(1-\frac{3}{p})} \|u^h\|_{\widetilde{L}^1_{t,f_1}(B^{-1+\frac{3}{p}}_{p,1})}.
\end{aligned}
$$

For $1 < p < 2$, we get by applying Lemma 2.1 that

$$
\begin{aligned}
&\big\|\Delta_j\big(R\big(u^3, \operatorname{div}_h u^h\big)\big)\big\|_{L^1_t(L^p)} \\
&\quad \lesssim 2^{3j(1-\frac{1}{p})} \sum_{j' \geqslant j-N_0} 2^{j'} \int_0^t \big\|\widetilde{\Delta}_{j'} u^3(t')\big\|_{L^{\frac{p}{p-1}}} \big\|\Delta_{j'} u^h(t')\big\|_{L^p}\, dt' \\
&\quad \lesssim 2^{3j(1-\frac{1}{p})} \sum_{j' \geqslant j-N_0} 2^{-\frac{3j'}{p}} 2^{3j'(\frac{2}{p}-1)} \int_0^t \big\|u^3(t')\big\|_{B^{1+\frac{3}{p}}_{p,1}} \big\|\Delta_{j'} u^h(t')\big\|_{L^p}\, dt' \\
&\quad \lesssim 2^{3j(1-\frac{1}{p})} \sum_{j' \geqslant j-N_0} d_{j'} 2^{-2j'} \|u^h\|_{\widetilde{L}^1_{t,f_1}(B^{-1+\frac{3}{p}}_{p,1})} \\
&\quad \lesssim d_j 2^{j(1-\frac{3}{p})} \|u^h\|_{\widetilde{L}^1_{t,f_1}(B^{-1+\frac{3}{p}}_{p,1})}.
\end{aligned}
$$

Along with (4.5), we prove the first inequality of Lemma 4.2.

On the other hand, it is easy to observe that

$$
\begin{aligned}
&\big\|\Delta_j\big(T_{u^3} \operatorname{div}_h u^h + T_{\operatorname{div}_h u^h} u^3\big)\big\|_{L^1_t(L^p)} \\
&\lesssim \sum_{|j'-j| \leqslant 5} \big(\|S_{j'-1} u^3\|_{L^\infty_t(L^\infty)} \|\Delta_{j'} \operatorname{div}_h u^h\|_{L^1_t(L^p)} + \|S_{j'-1} \operatorname{div}_h u^h\|_{L^\infty_t(L^\infty)} \|\Delta_{j'} u^3\|_{L^1(L^p)}\big) \\
&\lesssim d_j 2^{j(1-\frac{3}{p})} \big(\|u^3\|_{\widetilde{L}^\infty_t(B^{-1+\frac{3}{p}}_{p,1})} \|u^h\|_{L^1_t(B^{1+\frac{3}{p}}_{p,1})} + \|u^h\|_{\widetilde{L}^\infty_t(B^{-1+\frac{3}{p}}_{p,1})} \|u^3\|_{L^1_t(B^{1+\frac{3}{p}}_{p,1})}\big).
\end{aligned}
$$

Whereas again for $2 \leqslant p < 6$, we get by applying Lemma 2.1 that

$$
\begin{aligned}
\big\|\Delta_j\big(R\big(u^3, \operatorname{div}_h u^h\big)\big)\big\|_{L^1_t(L^p)} &\lesssim 2^{\frac{3j}{p}} \sum_{j' \geqslant j-N_0} 2^{j'} \|\Delta_{j'} u^3\|_{L^1_t(L^p)} \|\widetilde{\Delta}_{j'} u^h\|_{L^\infty_t(L^p)} \\
&\lesssim 2^{\frac{3j}{p}} \sum_{j' \geqslant j-N_0} d_{j'} 2^{j'(1-\frac{6}{p})} \|u^h\|_{\widetilde{L}^\infty_t(B^{-1+\frac{3}{p}}_{p,1})} \|u^3\|_{L^1_t(B^{1+\frac{3}{p}}_{p,1})} \\
&\lesssim d_j 2^{j(1-\frac{3}{p})} \|u^h\|_{\widetilde{L}^\infty_t(B^{-1+\frac{3}{p}}_{p,1})} \|u^3\|_{L^1_t(B^{1+\frac{3}{p}}_{p,1})}.
\end{aligned}
$$

The case $1 < p < 2$ follows the same lines as above. Whence thanks to (4.5), we obtain the second inequality of the lemma. $\square$

Lemma 4.3. *Let $p \geqslant q \geqslant 1$ with $\frac{1}{p}+\frac{1}{q}>\frac{1}{3}$ and f_1 be given by* (4.2)*. Then under the assumption of Proposition* 4.1*, we have*

$$\left\|\Delta_j\left(a\Delta u^3\right)\right\|_{L^1_t(L^p)} \lesssim d_j 2^{j(1-\frac{3}{p})}\|a\|_{\widetilde{L}^1_{t,f_1}(B^{\frac{3}{q}}_{q,1})}.$$

Proof. Again we first get by applying Bony's decomposition that

$$a\Delta u^3 = T_a\Delta u^3 + T_{\Delta u^3}a + R\left(a, \Delta u^3\right). \tag{4.6}$$

Applying Lemma 2.1 gives

$$\begin{aligned}\left\|\Delta_j\left(T_a\Delta u^3\right)(t')\right\|_{L^p} &\lesssim \sum_{|j'-j|\leqslant 5}\left\|S_{j'-1}a(t')\right\|_{L^\infty}\left\|\Delta_{j'}\Delta u^3(t')\right\|_{L^p}\\ &\lesssim d_j(t')2^{j(1-\frac{3}{p})}\sum_{\ell\in\mathbb{Z}}2^{\frac{3\ell}{q}}\left\|u^3(t')\right\|_{B^{1+\frac{3}{p}}_{p,1}}\left\|\Delta_\ell a(t')\right\|_{L^q},\end{aligned}$$

which implies that

$$\begin{aligned}\left\|T_a\left(\Delta u^3\right)\right\|_{L^1_t(B^{-1+\frac{3}{p}}_{p,1})} &\lesssim \sum_{j\in\mathbb{Z}}\int_0^t d_j(t')\sum_{\ell\in\mathbb{Z}}2^{\frac{3\ell}{q}}\left\|u^3(t')\right\|_{B^{1+\frac{3}{p}}_{p,1}}\left\|\Delta_\ell a(t')\right\|_{L^q}\,dt'\\ &\lesssim \sum_{\ell\in\mathbb{Z}}2^{\frac{3\ell}{q}}\int_0^t\left\|u^3(t')\right\|_{B^{1+\frac{3}{p}}_{p,1}}\left\|\Delta_\ell a(t')\right\|_{L^q}\,dt' \lesssim \|a\|_{\widetilde{L}^1_{t,f_1}(B^{\frac{3}{q}}_{q,1})}.\end{aligned}$$

While as $p \geqslant q$, we get by using Lemma 2.1 again that

$$\begin{aligned}\left\|\Delta_j(T_{\Delta u^3}a)\right\|_{L^1_t(L^p)} &\lesssim \sum_{|j'-j|\leqslant 5}2^{3j'(\frac{1}{q}-\frac{1}{p})}\int_0^t\left\|S_{j'-1}\Delta u^3(t')\right\|_{L^\infty}\left\|\Delta_{j'}a(t')\right\|_{L^q}\,dt'\\ &\lesssim 2^{j(1+3(\frac{1}{q}-\frac{1}{p}))}\sum_{|j'-j|\leqslant 5}\int_0^t\left\|u^3(t')\right\|_{B^{1+\frac{3}{p}}_{p,1}}\left\|\Delta_{j'}a(t')\right\|_{L^q}\,dt'\\ &\lesssim d_j 2^{j(1-\frac{3}{p})}\|a\|_{\widetilde{L}^1_{t,f_1}(B^{\frac{3}{q}}_{q,1})}.\end{aligned}$$

A similar procedure ensures for $\frac{1}{p}+\frac{1}{q}\leqslant 1$, that

$$\begin{aligned}\left\|\Delta_j\left(R\left(a, \Delta u^3\right)\right)(t')\right\|_{L^p} &\lesssim 2^{\frac{3j}{q}}\sum_{j'\geqslant j-N_0}\left\|\Delta_{j'}a(t')\right\|_{L^q}\left\|\widetilde{\Delta}_{j'}\Delta u^3(t')\right\|_{L^p}\\ &\lesssim 2^{\frac{3j}{q}}\sum_{j'\geqslant j-N_0}2^{j'(1-\frac{3}{p})}\left\|u^3(t')\right\|_{B^{1+\frac{3}{p}}_{p,1}}\left\|\Delta_{j'}a(t')\right\|_{L^q},\end{aligned}$$

integrating the above inequality over $[0,t]$ and using the fact that $\frac{1}{p}+\frac{1}{q}>\frac{1}{3}$, we arrive at

$$
\begin{aligned}
\big\|\Delta_j\big(R\big(a,\Delta u^3\big)\big)\big\|_{L^1_t(L^p)} &\lesssim 2^{\frac{3j}{q}}\sum_{j'\geqslant j-N_0} 2^{j'(1-\frac{3}{p})}\int_0^t \big\|u^3\big(t'\big)\big\|_{B^{1+\frac{3}{p}}_{p,1}}\big\|\Delta_{j'}a\big(t'\big)\big\|_{L^q}\,dt' \\
&\lesssim 2^{\frac{3j}{q}}\sum_{j'\geqslant j-N_0} d_{j'}2^{j'(1-\frac{3}{p}-\frac{3}{q})}\|a\|_{\widetilde{L}^1_{t,f_1}(B^{\frac{3}{q}}_{q,1})} \\
&\lesssim d_j 2^{j(1-\frac{3}{p})}\|a\|_{\widetilde{L}^1_{t,f_1}(B^{\frac{3}{q}}_{q,1})},
\end{aligned}
$$

which completes the proof of the lemma. The case when $\frac{1}{p}+\frac{1}{q}>1$ can be treated as in Lemma 2.2, we omit the details here. □

Now we are in a position to complete the proof of Proposition 4.1.

Proof of Proposition 4.1. Again as both the proof of the existence and uniqueness of solutions to (4.1) is essentially followed by the estimates (4.3) for some appropriate approximate solutions of (4.1). For simplicity, we just prove (4.3) for smooth enough solutions of (4.1). Indeed thanks to (4.1) and $\operatorname{div} u=0$, we have

$$
\begin{aligned}
\nabla\Pi_{\vec\lambda} = \nabla(-\Delta)^{-1}\big[&\operatorname{div}(a\nabla\Pi_{\vec\lambda}) + \operatorname{div}_h\operatorname{div}_h\big(u^h\otimes u^h_{\vec\lambda}\big) + 2\partial_3\operatorname{div}_h\big(u^3u^h_{\vec\lambda}\big) \\
&- 2\partial_3\big(u^3\operatorname{div}_h u^h_{\vec\lambda}\big) - \mu\operatorname{div}_h\big(a\Delta u^h_{\vec\lambda}\big) - \mu\partial_3\big(a_{\vec\lambda}\Delta u^3\big)\big].
\end{aligned}
$$

Applying Δ_j to the above equation and using Lemma 2.1 leads to

$$
\begin{aligned}
\big\|\Delta_j(\nabla\Pi_{\vec\lambda})\big\|_{L^1_t(L^p)} \lesssim{}& \big\|\Delta_j(a\nabla\Pi_{\vec\lambda})\big\|_{L^1_t(L^p)} + 2^j\big(\big\|\Delta_j\big(u^h\otimes u^h_{\vec\lambda}\big)\big\|_{L^1_t(L^p)} + \big\|\Delta_j\big(u^3u^h_{\vec\lambda}\big)\big\|_{L^1_t(L^p)}\big) \\
&+ \big\|\Delta_j\big(u^3\operatorname{div}_h u^h_{\vec\lambda}\big)\big\|_{L^1_t(L^p)} + \mu\big\|\Delta_j\big(a\Delta u^h_{\vec\lambda}\big)\big\|_{L^1_t(L^p)} \\
&+ \mu\big\|\Delta_j\big(a_{\vec\lambda}\Delta u^3\big)\big\|_{L^1_t(L^p)}.
\end{aligned}
\tag{4.7}
$$

However as $p\geqslant q$ and $\frac{1}{p}+\frac{1}{q}>\frac{1}{3}$, applying Lemma 2.2 and standard product laws in Besov space gives rise to

$$
\begin{aligned}
&\big\|\Delta_j(a\nabla\Pi_{\vec\lambda})\big\|_{L^1_t(L^p)} \lesssim d_j 2^{j(1-\frac{3}{p})}\|a\|_{L^\infty_t(B^{\frac{3}{q}}_{q,1})}\|\nabla\Pi_{\vec\lambda}\|_{L^1_t(B^{-1+\frac{3}{p}}_{p,1})} \quad\text{and} \\
&\big\|\Delta_j\big(u^h\otimes u^h_{\vec\lambda}\big)\big\|_{L^1_t(L^p)} \lesssim d_j 2^{-\frac{3j}{p}}\big\|u^h\big\|_{L^\infty_t(B^{-1+\frac{3}{p}}_{p,1})}\big\|u^h_{\vec\lambda}\big\|_{L^1_t(B^{1+\frac{3}{p}}_{p,1})}, \\
&\big\|\Delta_j\big(a\Delta u^h_{\vec\lambda}\big)\big\|_{L^1_t(L^p)} \lesssim d_j 2^{j(1-\frac{3}{p})}\|a\|_{L^\infty_t(B^{\frac{3}{q}}_{q,1})}\big\|u^h_{\vec\lambda}\big\|_{L^1_t(B^{1+\frac{3}{p}}_{p,1})},
\end{aligned}
\tag{4.8}
$$

which along with Lemma 4.1 to Lemma 4.3 and (4.7) implies that

$$\begin{aligned}\|\Delta_j(\nabla\Pi_{\vec\lambda})\|_{L^1_t(L^p)} \lesssim d_j 2^{j(1-\frac{3}{p})}\Big\{&\|a\|_{L^\infty_t(B^{\frac{3}{q}}_{q,1})}\|\nabla\Pi_{\vec\lambda}\|_{L^1_t(B^{-1+\frac{3}{p}}_{p,1})}\\
&+\big(\mu\|a\|_{L^\infty_t(B^{\frac{3}{q}}_{q,1})}+\|u^h\|_{L^\infty_t(B^{-1+\frac{3}{p}}_{p,1})}\big)\|u^h_{\vec\lambda}\|_{L^1_t(B^{1+\frac{3}{p}}_{p,1})}\\
&+\|u^h_{\vec\lambda}\|^{\frac12}_{L^1_t(B^{1+\frac{3}{p}}_{p,1})}\|u^h_{\vec\lambda}\|^{\frac12}_{\widetilde{L}^1_{t,f_2}(B^{-1+\frac{3}{p}}_{p,1})}\\
&+\|u^h_{\vec\lambda}\|_{\widetilde{L}^1_{t,f_1}(B^{-1+\frac{3}{p}}_{p,1})}+\mu\|a_{\vec\lambda}\|_{\widetilde{L}^1_{t,f_1}(B^{\frac{3}{q}}_{q,1})}\Big\},\end{aligned}$$

from which, we deduce that

$$\begin{aligned}\|\nabla\Pi_{\vec\lambda}\|_{L^1_t(B^{-1+\frac{3}{p}}_{p,1})} \leqslant C\Big\{&\|a\|_{L^\infty_t(B^{\frac{3}{q}}_{q,1})}\|\nabla\Pi_{\vec\lambda}\|_{L^1_t(B^{-1+\frac{3}{p}}_{p,1})}\\
&+\big(\mu\|a\|_{L^\infty_t(B^{\frac{3}{q}}_{q,1})}+\|u^h\|_{L^\infty_t(B^{-1+\frac{3}{p}}_{p,1})}\big)\|u^h_{\vec\lambda}\|_{L^1_t(B^{1+\frac{3}{p}}_{p,1})}\\
&+\|u^h_{\vec\lambda}\|^{\frac12}_{L^1_t(B^{1+\frac{3}{p}}_{p,1})}\|u^h_{\vec\lambda}\|^{\frac12}_{\widetilde{L}^1_{t,f_2}(B^{-1+\frac{3}{p}}_{p,1})}\\
&+\|u^h_{\vec\lambda}\|_{\widetilde{L}^1_{t,f_1}(B^{-1+\frac{3}{p}}_{p,1})}+\mu\|a_{\vec\lambda}\|_{\widetilde{L}^1_{t,f_1}(B^{\frac{3}{q}}_{q,1})}\Big\} \quad \text{for } t\leqslant T.\end{aligned}$$

This in particular implies (4.3) if $C\|a\|_{L^\infty_t(B^{\frac{3}{q}}_{q,1})}\leqslant\frac12$. This finishes the proof of Proposition 4.1. □

To deal with the estimate of u^3, we also need the following proposition:

Proposition 4.2. *Under the assumptions of Proposition* 4.1, *one has*

$$\begin{aligned}\|\nabla\Pi\|_{L^1_t(B^{-1+\frac{3}{p}}_{p,1})} \leqslant \frac{C}{1-C\|a\|_{L^\infty_t(B^{\frac{3}{q}}_{q,1})}}\Big\{&\|u^h\|_{L^1_t(B^{1+\frac{3}{p}}_{p,1})}\|u^3\|_{\widetilde{L}^\infty_t(B^{-1+\frac{3}{p}}_{p,1})}\\
&+\big(\|u^h\|_{L^1_t(B^{1+\frac{3}{p}}_{p,1})}+\|u^3\|_{L^1_t(B^{1+\frac{3}{p}}_{p,1})}\big)\\
&\times\big(\mu\|a\|_{\widetilde{L}^\infty_t(B^{\frac{3}{q}}_{q,1})}+\|u^h\|_{\widetilde{L}^\infty_t(B^{-1+\frac{3}{p}}_{p,1})}\big)\Big\}\end{aligned}\tag{4.9}$$

for $t\leqslant T$ provided that $C\|a\|_{L^\infty_t(B^{\frac{3}{q}}_{q,1})}\leqslant\frac12$.

Proof. The proof of this proposition follows exactly the same lines as that of Proposition 4.1. Indeed taking $\lambda_1=\lambda_2=0$ in (4.7), (4.8) and applying Lemma 4.1 to Lemma 4.3, we arrive at

$$\|\nabla\Pi\|_{L^1_t(B^{-1+\frac{3}{p}}_{p,1})} \leqslant C\big\{\|a\|_{L^\infty_t(B^{\frac{3}{q}}_{q,1})}\|\nabla\Pi\|_{L^1_t(B^{-1+\frac{3}{p}}_{p,1})} + \|u^h\|_{L^1_t(B^{1+\frac{3}{p}}_{p,1})}\|u^3\|_{\widetilde{L}^\infty_t(B^{-1+\frac{3}{p}}_{p,1})}$$
$$+ \big(\|u^h\|_{L^1_t(B^{1+\frac{3}{p}}_{p,1})} + \|u^3\|_{L^1_t(B^{1+\frac{3}{p}}_{p,1})}\big)\big(\mu\|a\|_{\widetilde{L}^\infty_t(B^{\frac{3}{q}}_{q,1})} + \|u^h\|_{\widetilde{L}^\infty_t(B^{-1+\frac{3}{p}}_{p,1})}\big)\big\}$$

for $t \leqslant T$, from which and the fact that $C\|a\|_{L^\infty_t(B^{\frac{3}{q}}_{q,1})} \leqslant \frac{1}{2}$, we conclude the proof of (4.9). □

5. The proof of Theorem 1.2

The goal of this section is to present the proof of Theorem 1.2. Indeed given $a_0 \in B^{\frac{3}{q}}_{q,1}(\mathbb{R}^3)$, $u_0 \in B^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)$ with $\|a_0\|_{B^{\frac{3}{q}}_{q,1}}$ sufficiently small and p, q satisfying the conditions listed in Theorem 1.2, Theorem 1.1 and Theorem 2 of [12] ensure that there exists a positive time T so that (1.3) has a unique solution (a, u, Π) with

$$a \in \mathcal{C}\big([0,T]; B^{\frac{3}{q}}_{q,1}\big(\mathbb{R}^3\big)\big), \qquad u \in \mathcal{C}\big([0,T]; B^{-1+\frac{3}{p}}_{p,1}\big(\mathbb{R}^3\big)\big) \cap L^1\big((0,T); B^{1+\frac{3}{p}}_{p,1}\big(\mathbb{R}^3\big)\big) \quad \text{and}$$
$$\nabla\Pi \in L^1\big((0,T); B^{-1+\frac{3}{p}}_{p,1}\big(\mathbb{R}^3\big)\big). \tag{5.1}$$

We denote T^* to be the largest time so that there holds (5.1). Hence to prove Theorem 1.2, we only need to prove that $T^* = \infty$ and there holds (1.5). Toward this and motivated by [19,23,15], we shall deal with the L^p type energy estimate for u^h and u^3 separately.

5.1. *The estimate of* u^h

As in Proposition 4.1, let $f_1(t)$, $f_2(t)$, $a_{\vec{\lambda}}$, $u_{\vec{\lambda}}$, $\Pi_{\vec{\lambda}}$ be given by (4.2), and

$$a_{\lambda_1} \stackrel{\text{def}}{=} a\exp\left(-\lambda_1\int_0^t f_1(t')\,dt'\right).$$

Then thanks to (1.3), we have

$$\partial_t u^h_{\vec{\lambda}} + \big(\lambda_1 f_1(t) + \lambda_2 f_2(t)\big)u_{\vec{\lambda}} - \mu\Delta u^h_{\vec{\lambda}} = -u\cdot\nabla u^h_{\vec{\lambda}} - (1+a)\nabla_h\Pi_{\vec{\lambda}} + \mu a_{\vec{\lambda}}\Delta u^h.$$

Applying the operator Δ_j to the above equation and taking the L^2 inner product of the resulting equation with $|\Delta_j u^h_{\vec{\lambda}}|^{p-2}\Delta_j u^h_{\vec{\lambda}}$ (again in the case when $p \in (1,2)$, we need to make some modification as that [9]), we obtain

$$\frac{1}{p}\frac{d}{dt}\big\|\Delta_j u^h_{\vec{\lambda}}(t)\big\|^p_{L^p} + \big(\lambda_1 f_1(t) + \lambda_2 f_2(t)\big)\big\|\Delta_j u^h_{\vec{\lambda}}(t)\big\|^p_{L^p} - \mu\int_{\mathbb{R}^3}\Delta\Delta_j u^h_{\vec{\lambda}} \mid \big|\Delta_j u^h_{\vec{\lambda}}\big|^{p-2}\Delta_j u^h_{\vec{\lambda}}\,dx$$
$$= -\int_{\mathbb{R}^3}\big(\Delta_j\big(u\cdot\nabla u^h_{\vec{\lambda}}\big) + \Delta_j\big((1+a)\nabla_h\Pi_{\vec{\lambda}}\big) - \mu\Delta_j\big(a_{\vec{\lambda}}\Delta u^h\big)\big) \mid \big|\Delta_j u^h_{\vec{\lambda}}\big|^{p-2}\Delta_j u^h_{\vec{\lambda}}\,dx.$$

However thanks to [9,21], there exists a positive constant $\bar c$ so that

$$-\int_{\mathbb{R}^3} \Delta\Delta_j u^h_{\bar\lambda} \,\big|\,\big|\Delta_j u^h_{\bar\lambda}\big|^{p-2}\Delta_j u^h_{\bar\lambda}\,dx \geqslant \bar c 2^{2j}\big\|\Delta_j u^h_{\bar\lambda}\big\|^p_{L^p},$$

whence a similar argument as that in [9] gives rise to

$$\begin{aligned}
&\frac{d}{dt}\big\|\Delta_j u^h_{\bar\lambda}(t)\big\|_{L^p} + \big(\lambda_1 f_1(t)+\lambda_2 f_2(t)\big)\big\|\Delta_j u^h_{\bar\lambda}(t)\big\|_{L^p} + \bar c\mu 2^{2j}\big\|\Delta_j u^h_{\bar\lambda}(t)\big\|_{L^p}\\
&\quad\leqslant \big\|\Delta_j\big(u\cdot\nabla u^h_{\bar\lambda}\big)\big\|_{L^p} + \big\|\Delta_j\big((1+a)\nabla_h \Pi_{\bar\lambda}\big)\big\|_{L^p} + \mu\big\|\Delta_j\big(a_{\bar\lambda}\Delta u^h\big)\big\|_{L^p}.
\end{aligned}\tag{5.2}$$

Applying Lemma 2.2 and Lemma 4.1, we obtain

$$\begin{aligned}
\big\|\Delta_j\big(u\cdot\nabla u^h_{\bar\lambda}\big)\big\|_{L^1_t(L^p)} &\leqslant 2^j\big(\big\|\Delta_j\big(u^h\otimes u^h_{\bar\lambda}\big)\big\|_{L^1_t(L^p)} + \big\|\Delta_j\big(u^3 u^h_{\bar\lambda}\big)\big\|_{L^1_t(L^p)}\big)\\
&\leqslant Cd_j 2^{j(1-\frac{3}{p})}\big(\big\|u^h\big\|_{\widetilde L^\infty_t(B^{-1+\frac{3}{p}}_{p,1})}\big\|u^h_{\bar\lambda}\big\|_{L^1_t(B^{1+\frac{3}{p}}_{p,1})}\\
&\quad + \big\|u^h_{\bar\lambda}\big\|^{\frac12}_{L^1_t(B^{1+\frac{3}{p}}_{p,1})}\big\|u^h_{\bar\lambda}\big\|^{\frac12}_{\widetilde L^1_{t,f_2}(B^{-1+\frac{3}{p}}_{p,1})} + \big\|u^h_{\bar\lambda}\big\|_{\widetilde L^1_{t,f_1}(B^{-1+\frac{3}{p}}_{p,1})}\big).
\end{aligned}$$

While applying Lemma 2.2 leads to

$$\big\|\Delta_j\big(a_{\bar\lambda}\Delta u^h\big)\big\|_{L^1_t(L^p)} \leqslant Cd_j 2^{j(1-\frac{3}{p})}\|a_{\bar\lambda}\|_{\widetilde L^\infty_t(B^{\frac{3}{q}}_{q,1})}\big\|u^h\big\|_{L^1_t(B^{1+\frac{3}{p}}_{p,1})},$$

and

$$\big\|\Delta_j\big((1+a)\nabla_h\Pi_{\bar\lambda}\big)\big\|_{L^1_t(L^p)} \leqslant Cd_j 2^{j(1-\frac{3}{p})}\big(1+\|a\|_{\widetilde L^\infty_t(B^{\frac{3}{q}}_{q,1})}\big)\|\nabla_h\Pi_{\bar\lambda}\|_{L^1_t(B^{-1+\frac{3}{p}}_{p,1})}.$$

Moreover under the assumption that

$$\|a\|_{\widetilde L^\infty_T(B^{\frac{3}{q}}_{q,1})} \leqslant c_1 \tag{5.3}$$

for some c_1 sufficiently small, (4.3) ensures that

$$\begin{aligned}
\big\|\Delta_j\big((1+a)\nabla_h\Pi_{\bar\lambda}\big)\big\|_{L^1_t(L^p)} &\leqslant Cd_j 2^{j(1-\frac{3}{p})}\big\{\big(\mu\|a\|_{L^\infty_t(B^{\frac{3}{q}}_{q,1})} + \big\|u^h\big\|_{\widetilde L^\infty_t(B^{-1+\frac{3}{p}}_{p,1})}\big)\big\|u^h_{\bar\lambda}\big\|_{L^1_t(B^{1+\frac{3}{p}}_{p,1})}\\
&\quad + \big\|u^h_{\bar\lambda}\big\|^{\frac12}_{L^1_t(B^{1+\frac{3}{p}}_{p,1})}\big\|u^h_{\bar\lambda}\big\|^{\frac12}_{\widetilde L^1_{t,f_2}(B^{-1+\frac{3}{p}}_{p,1})} + \big\|u^h_{\lambda}\big\|_{\widetilde L^1_{t,f_1}(B^{-1+\frac{3}{p}}_{p,1})}\\
&\quad + \mu\|a_{\bar\lambda}\|_{\widetilde L^1_{t,f_1}(B^{\frac{3}{q}}_{q,1})}\big\}.
\end{aligned}$$

Integrating (5.2) over $[0,t]$ and substituting the above estimates into the resulting inequality, we obtain

$$\begin{aligned}
&\|u^h_{\bar\lambda}\|_{\widetilde{L}^\infty_t(B^{-1+\frac{3}{p}}_{p,1})}+\lambda_1\|u^h_{\bar\lambda}\|_{\widetilde{L}^1_{t,f_1}(B^{-1+\frac{3}{p}}_{p,1})}+\lambda_2\|u^h_{\bar\lambda}\|_{\widetilde{L}^1_{t,f_2}(B^{-1+\frac{3}{p}}_{p,1})}+\bar c\mu\|u^h_{\bar\lambda}\|_{L^1_t(B^{1+\frac{3}{p}}_{p,1})}\\
&\leqslant\|u^h_0\|_{B^{-1+\frac{3}{p}}_{p,1}}+\frac{\bar c\mu}{2}\|u^h_{\bar\lambda}\|_{L^1_t(B^{1+\frac{3}{p}}_{p,1})}+C\Big\{\big(\mu\|a\|_{L^\infty_t(B^{\frac{3}{q}}_{q,1})}+\|u^h\|_{\widetilde{L}^\infty_t(B^{-1+\frac{3}{p}}_{p,1})}\big)\|u^h_{\bar\lambda}\|_{L^1_t(B^{1+\frac{3}{p}}_{p,1})}\\
&\quad+\frac{1}{\mu}\|u^h_{\bar\lambda}\|_{\widetilde{L}^1_{t,f_2}(B^{-1+\frac{3}{p}}_{p,1})}+\|u^h_{\bar\lambda}\|_{\widetilde{L}^1_{t,f_1}(B^{-1+\frac{3}{p}}_{p,1})}+\mu\|a_{\bar\lambda}\|_{\widetilde{L}^1_{t,f_1}(B^{\frac{3}{q}}_{q,1})}\Big\}\quad\text{for}\quad t\leqslant T,
\end{aligned}\tag{5.4}$$

under the assumption of (5.3).

5.2. The estimate of u^3

We use that the equation on the vertical component of the velocity is a linear equation with coefficients depending on the horizontal components and a. Thanks to the u^3 equation of (1.3), we get by a similar derivation of (5.2) that

$$\begin{aligned}
&\|\Delta_j u^3\|_{L^\infty_t(L^p)}+\bar c\mu 2^{2j}\|\Delta_j u^3\|_{L^1_t(L^p)}\\
&\quad\leqslant\|\Delta_j u^3_0\|_{L^p}+C\big(\|\Delta_j(u\cdot\nabla u^3)\|_{L^1_t(L^p)}+\|\Delta_j((1+a)\partial_3\Pi)\|_{L^1_t(L^p)}\\
&\qquad+\mu\|\Delta_j(a\Delta u^3)\|_{L^1_t(L^p)}\big).
\end{aligned}\tag{5.5}$$

Applying Lemma 4.1 and Lemma 4.2 ensures that

$$\begin{aligned}
\|\Delta_j(u\cdot\nabla u^3)\|_{L^1_t(L^p)}&\lesssim 2^j\|\Delta_j(u^hu^3)\|_{L^1_t(L^p)}+\|\Delta_j(u^3\operatorname{div}_h u^h)\|_{L^1_t(L^p)}\\
&\lesssim d_j2^{j(1-\frac{3}{p})}\big(\|u^h\|_{\widetilde{L}^\infty_t(B^{-1+\frac{3}{p}}_{p,1})}\|u^3\|_{L^1_t(B^{1+\frac{3}{p}}_{p,1})}\\
&\quad+\|u^h\|_{L^1_t(B^{1+\frac{3}{p}}_{p,1})}\|u^3\|_{\widetilde{L}^\infty_t(B^{-1+\frac{3}{p}}_{p,1})}\big).
\end{aligned}$$

Whereas under the assumption of (5.3), we get by applying Lemma 2.2 and Proposition 4.2 that

$$\begin{aligned}
\|\Delta_j((1+a)\partial_3\Pi)\|_{L^1_t(L^p)}&\leqslant Cd_j2^{j(1-\frac{3}{p})}\big(1+\|a\|_{\widetilde{L}^\infty_t(B^{\frac{3}{q}}_{q,1})}\big)\|\partial_3\Pi\|_{L^1_t(B^{-1+\frac{3}{p}}_{p,1})}\\
&\leqslant Cd_j2^{j(1-\frac{3}{p})}\big\{\|u^h\|_{L^1_t(B^{1+\frac{3}{p}}_{p,1})}\|u^3\|_{\widetilde{L}^\infty_t(B^{-1+\frac{3}{p}}_{p,1})}\\
&\quad+\big(\|u^h\|_{L^1_t(B^{1+\frac{3}{p}}_{p,1})}+\|u^3\|_{L^1_t(B^{1+\frac{3}{p}}_{p,1})}\big)\\
&\quad\times\big(\mu\|a\|_{\widetilde{L}^\infty_t(B^{\frac{3}{q}}_{q,1})}+\|u^h\|_{\widetilde{L}^\infty_t(B^{-1+\frac{3}{p}}_{p,1})}\big)\big\}.
\end{aligned}$$

Then we get by substituting the above estimates into (5.5) that

$$\begin{aligned}
&\|u^3\|_{\widetilde{L}^\infty_t(B^{-1+\frac{3}{p}}_{p,1})} + \bar{c}\mu\|u^3\|_{L^1_t(B^{1+\frac{3}{p}}_{p,1})} \\
&\quad\leqslant \|u^3_0\|_{B^{-1+\frac{3}{p}}_{p,1}} + C\big\{\|u^h\|_{L^1_t(B^{1+\frac{3}{p}}_{p,1})}\|u^3\|_{\widetilde{L}^\infty_t(B^{-1+\frac{3}{p}}_{p,1})} \\
&\qquad + \big(\|u^h\|_{L^1_t(B^{1+\frac{3}{p}}_{p,1})} + \|u^3\|_{L^1_t(B^{1+\frac{3}{p}}_{p,1})}\big)\big(\mu\|a\|_{\widetilde{L}^\infty_t(B^{\frac{3}{q}}_{q,1})} + \|u^h\|_{\widetilde{L}^\infty_t(B^{-1+\frac{3}{p}}_{p,1})}\big)\big\}. \qquad (5.6)
\end{aligned}$$

5.3. The proof of Theorem 1.2

With (5.4) and (5.6), we can prove that $T^* = \infty$ provided that there holds (1.4). In fact, we first get by taking $\lambda = \lambda_1$ in Proposition 3.1 that

$$\|a_{\lambda_1}\|_{\widetilde{L}^\infty_t(B^{\frac{3}{q}}_{q,1})} + \frac{\lambda_1}{2}\|a_{\lambda_1}\|_{\widetilde{L}^1_{t,f_1}(B^{\frac{3}{q}}_{q,1})} \leqslant \|a_0\|_{B^{\frac{3}{q}}_{q,1}} + C\|u^h\|_{L^1_t(B^{1+\frac{3}{p}}_{p,1})}\|a_{\lambda_1}\|_{\widetilde{L}^\infty_t(B^{\frac{3}{q}}_{q,1})}. \qquad (5.7)$$

While thanks to Definition 1.1 and (4.2), one has

$$\|a_{\vec{\lambda}}\|_{\widetilde{L}^1_{t,f_1}(B^{\frac{3}{q}}_{q,1})} \leqslant \|a_{\lambda_1}\|_{\widetilde{L}^1_{t,f_1}(B^{\frac{3}{q}}_{q,1})}.$$

Then taking $\lambda_1 \geqslant 4C$ and $\lambda_2 \geqslant \frac{2C}{\mu}$, we get by summing up (5.4) and (5.7) $\times\ \mu$ that

$$\begin{aligned}
&\mu\|a_{\lambda_1}\|_{\widetilde{L}^\infty_t(B^{\frac{3}{q}}_{q,1})} + \|u^h_{\vec{\lambda}}\|_{\widetilde{L}^\infty_t(B^{-1+\frac{3}{p}}_{p,1})} + \frac{\lambda_1}{2}\bigg(\frac{\mu}{2}\|a_\lambda\|_{\widetilde{L}^1_{t,f_1}(B^{\frac{3}{q}}_{q,1})} + \|u^h_{\vec{\lambda}}\|_{\widetilde{L}^1_{t,f_1}(B^{-1+\frac{3}{p}}_{p,1})}\bigg) \\
&\quad + \frac{\lambda_2}{2}\|u^h_{\vec{\lambda}}\|_{\widetilde{L}^1_{t,f_2}(B^{-1+\frac{3}{p}}_{p,1})} + \frac{\bar{c}\mu}{2}\|u^h_{\vec{\lambda}}\|_{L^1_t(B^{1+\frac{3}{p}}_{p,1})} \\
&\leqslant \mu\|a_0\|_{B^{\frac{3}{q}}_{q,1}} + \|u^h_0\|_{B^{-1+\frac{3}{p}}_{p,1}} + C_1\big\{\mu\|u^h\|_{L^1_t(B^{1+\frac{3}{p}}_{p,1})}\|a_{\lambda_1}\|_{\widetilde{L}^\infty_t(B^{\frac{3}{q}}_{q,1})} \\
&\quad + \big(\mu\|a\|_{\widetilde{L}^\infty_t(B^{\frac{3}{q}}_{q,1})} + \|u^h\|_{\widetilde{L}^\infty_t(B^{-1+\frac{3}{p}}_{p,1})}\big)\|u^h_{\vec{\lambda}}\|_{L^1_t(B^{1+\frac{3}{p}}_{p,1})}\big\} \quad \text{for } t \leqslant T, \qquad (5.8)
\end{aligned}$$

under the assumption of (5.3).

Now let c_2 be a small enough positive constant, which will be determined later on, we define $\mathfrak{T}$ by

$$\mathfrak{T} \stackrel{\text{def}}{=} \max\big\{t \in [0, T^*):\ \|u^h\|_{\widetilde{L}^\infty_t(B^{-1+\frac{3}{p}}_{p,1})} + \mu\big(\|a\|_{\widetilde{L}^\infty_t(B^{\frac{3}{q}}_{q,1})} + \|u^h\|_{L^1_t(B^{1+\frac{3}{p}}_{p,1})}\big) \leqslant c_2\mu\big\}. \qquad (5.9)$$

(5.9) implies that $\|a\|_{\widetilde{L}^\infty_{\mathfrak{T}}(B^{\frac{3}{q}}_{q,1})} \leqslant c_2$, in particular, if we take $c_2 \leqslant c_1$ in (5.3). Then there automatically holds (5.3) for $t \leqslant \mathfrak{T}$. In what follows, we shall prove that $\mathfrak{T} = \infty$ under the assumption of (1.4). Otherwise, taking $c_2 \leqslant \bar{c}_2 \stackrel{\text{def}}{=} \min(c_1, \frac{\bar{c}}{4C_1}, \frac{3}{4C_1})$, we deduce from (5.8) that

$$\|u^h_{\vec\lambda}\|_{\widetilde{L}^\infty_t(B^{-1+\frac{3}{p}}_{p,1})}+\frac{\mu}{4}\big(\|a_{\lambda_1}\|_{\widetilde{L}^\infty_t(B^{\frac{3}{q}}_{q,1})}+\bar c\|u^h_{\vec\lambda}\|_{L^1_t(B^{1+\frac{3}{p}}_{p,1})}\big)\leqslant\mu\|a_0\|_{B^{\frac{3}{q}}_{q,1}}+\|u^h_0\|_{B^{-1+\frac{3}{p}}_{p,1}},$$

for $t\leqslant\mathfrak{T}$.

On the other hand, it is easy to observe from (4.2) that

$$\begin{aligned}&\Big(\|u^h\|_{\widetilde{L}^\infty_t(B^{-1+\frac{3}{p}}_{p,1})}+\frac{\mu}{4}\big(\|a\|_{\widetilde{L}^\infty_t(B^{\frac{3}{q}}_{q,1})}+\bar c\|u^h\|_{L^1_t(B^{1+\frac{3}{p}}_{p,1})}\big)\Big)\exp\Big\{-\int_0^t(\lambda_1f_1+\lambda_2f_2)(t')\,dt'\Big\}\\&\quad\leqslant\|u^h_{\vec\lambda}\|_{\widetilde{L}^\infty_t(B^{-1+\frac{3}{p}}_{p,1})}+\frac{\mu}{4}\big(\|a_{\lambda_1}\|_{\widetilde{L}^\infty_t(B^{\frac{3}{q}}_{q,1})}+\|u^h_{\vec\lambda}\|_{L^1_t(B^{1+\frac{3}{p}}_{p,1})}\big).\end{aligned}$$

As a consequence, we obtain

$$\begin{aligned}&\|u^h\|_{\widetilde{L}^\infty_t(B^{-1+\frac{3}{p}}_{p,1})}+\frac{\mu}{4}\big(\|a\|_{\widetilde{L}^\infty_t(B^{\frac{3}{q}}_{q,1})}+\|u^h\|_{L^1_t(B^{1+\frac{3}{p}}_{p,1})}\big)\\&\quad\leqslant\big(\mu\|a_0\|_{B^{\frac{3}{q}}_{q,1}}+\|u^h_0\|_{B^{-1+\frac{3}{p}}_{p,1}}\big)\exp\Big\{4C\int_0^t\Big(\|u^3(t')\|_{B^{1+\frac{3}{p}}_{p,1}}+\frac{1}{\mu}\|u^3(t')\|^2_{B^{\frac{3}{p}}_{p,1}}\Big)\,dt'\Big\}\end{aligned}\tag{5.10}$$

for $t\leqslant\mathfrak{T}$.

While thanks to (5.6), we get by taking $c_2\leqslant\min\{\bar c_2,\frac{1}{2C},\frac{\bar c}{2C}\}$ that

$$\|u^3\|_{\widetilde{L}^\infty_t(B^{-1+\frac{3}{p}}_{p,1})}+\mu\bar c\|u^3\|_{L^1_t(B^{1+\frac{3}{p}}_{p,1})}\leqslant2\|u^3_0\|_{B^{-1+\frac{3}{p}}_{p,1}}+c_2\mu\tag{5.11}$$

for $t\leqslant\mathfrak{T}$.

Combing (5.10) with (5.11), we reach

$$\begin{aligned}&\|u^h\|_{\widetilde{L}^\infty_t(B^{-1+\frac{3}{p}}_{p,1})}+\frac{\mu}{4}\big(\|a\|_{\widetilde{L}^\infty_t(B^{\frac{3}{q}}_{q,1})}+\|u^h\|_{L^1_t(B^{1+\frac{3}{p}}_{p,1})}\big)\\&\quad\leqslant\mathfrak{C}_1\big(\mu\|a_0\|_{B^{\frac{3}{q}}_{q,1}}+\|u^h_0\|_{B^{-1+\frac{3}{p}}_{p,1}}\big)\exp\Big\{\frac{\mathfrak{C}_2}{\mu^2}\|u^3_0\|^2_{B^{-1+\frac{3}{p}}_{p,1}}\Big\}\end{aligned}\tag{5.12}$$

for $t\leqslant\mathfrak{T}$ and some positive constants $\mathfrak{C}_1$, $\mathfrak{C}_2$ which depends on $\bar c$ and c_2. In particular, (5.12) implies that if we take C_0 large enough and c_0 sufficiently small in (1.4), there holds

$$\|u^h\|_{\widetilde{L}^\infty_t(B^{-1+\frac{3}{p}}_{p,1})}+\mu\big(\|a\|_{\widetilde{L}^\infty_t(B^{\frac{3}{q}}_{q,1})}+\|u^h\|_{L^1_t(B^{1+\frac{3}{p}}_{p,1})}\big)\leqslant C\eta\leqslant\frac{c_2}{2}\mu\quad\text{for }t\leqslant\mathfrak{T},$$

which contradicts with (5.9). Whence we conclude that $\mathfrak{T}=\infty$, and there holds

$$\begin{aligned}&\|u^h\|_{\widetilde{L}^\infty(\mathbb{R}^+;B^{-1+\frac{3}{p}}_{p,1})}+\mu\big(\|a\|_{\widetilde{L}^\infty(\mathbb{R}^+;B^{\frac{3}{q}}_{q,1})}+\|u^h\|_{L^1(\mathbb{R}^+;B^{1+\frac{3}{p}}_{p,1})}\big)\leqslant C\eta,\\&\|u^3\|_{\widetilde{L}^\infty(\mathbb{R}^+;B^{-1+\frac{3}{p}}_{p,1})}+\mu\|u^3\|_{L^1(\mathbb{R}^+;B^{1+\frac{3}{p}}_{p,1})}\leqslant2\|u^3_0\|_{B^{-1+\frac{3}{p}}_{p,1}}+c_2\mu.\end{aligned}\tag{5.13}$$

Finally let us turn to the proof of (1.5). Indeed thanks to (1.3), we have

$$u = e^{\mu t\Delta}u_0 + \int_0^t e^{\mu(t-s)\Delta}\mathbb{P}\big\{-\mathrm{div}_h\big(u^h \otimes u\big) - \partial_3\big(u^3 u\big) + \mu a\Delta u - a\nabla\Pi\big\}(s)\,ds,$$

where $\mathbb{P}$ denotes the Leray projection operator to the divergence free vector field space. Then we get by applying a lemma from [6] on heat semi-group that

$$\begin{aligned}\big\|\Delta_j\big(u - e^{\mu t\Delta}u_0\big)(t)\big\|_{L^p} \leqslant C\int_0^t e^{-\mu(t-s)2^{2j}}\big(&2^j\big\|\Delta_j\big(u^h\otimes u\big)(s)\big\|_{L^p} + \big\|\Delta_j\partial_3\big(u^3u\big)(s)\big\|_{L^p}\\ &+ \mu\big\|\Delta_j(a\Delta u)(s)\big\|_{L^p} + \big\|\Delta_j(a\nabla\Pi)(s)\big\|_{L^p}\big)\,ds.\end{aligned} \tag{5.14}$$

Applying Lemma 4.1, Lemma 4.2 and (5.13) that

$$\begin{aligned}\big\|\Delta_j\partial_3\big(u^3u\big)\big\|_{L^1(L^p)} &\lesssim 2^j\big\|\Delta_j\big(u^3u^h\big)\big\|_{L^1_t(L^p)} + \big\|\Delta_j\big(u^3\,\mathrm{div}_h\,u^h\big)\big\|_{L^1_t(L^p)}\\ &\lesssim d_j 2^{j(1-\frac{3}{p})}\Big(\big\|u^h\big\|_{\widetilde{L}^\infty_t(B^{-1+\frac{3}{p}}_{p,1})}\big\|u^3\big\|_{L^1_t(B^{1+\frac{3}{p}}_{p,1})}\\ &\quad + \big\|u^h\big\|_{L^1_t(B^{1+\frac{3}{p}}_{p,1})}\big\|u^3\big\|_{\widetilde{L}^\infty_t(B^{-1+\frac{3}{p}}_{p,1})}\Big)\\ &\lesssim d_j 2^{j(1-\frac{3}{p})}\big(2\big\|u_0^3\big\|_{B^{-1+\frac{3}{p}}_{p,1}} + c_2\mu\big)\frac{\eta}{\mu}.\end{aligned}$$

While taking $\lambda_1 = \lambda_2 = 0$ in (4.8) leads to

$$\begin{aligned}&2^j\big\|\Delta_j\big(u^h\otimes u\big)\big\|_{L^1_t(L^p)} + \mu\big\|\Delta_j(a\Delta u)\big\|_{L^1_t(L^p)} + \big\|\Delta_j(a\nabla\Pi)\big\|_{L^1_t(L^p)}\\ &\quad\lesssim d_j 2^{j(1-\frac{3}{p})}\Big\{\Big(\big\|u^h\big\|_{\widetilde{L}^\infty_t(B^{-1+\frac{3}{p}}_{p,1})} + \mu\|a\|_{L^\infty_t(B^{\frac{3}{q}}_{q,1})}\Big)\|u\|_{L^1_t(B^{1+\frac{3}{p}}_{p,1})}\\ &\qquad + \big\|u^h\big\|_{L^1_t(B^{1+\frac{3}{p}}_{p,1})}\|u\|_{\widetilde{L}^\infty_t(B^{-1+\frac{3}{p}}_{p,1})} + \|a\|_{L^\infty_t(B^{\frac{3}{q}}_{q,1})}\|\nabla\Pi\|_{L^1_t(B^{-1+\frac{3}{p}}_{p,1})}\Big\}\\ &\quad\lesssim d_j 2^{j(1-\frac{3}{p})}\big(2\big\|u_0^3\big\|_{B^{-1+\frac{3}{p}}_{p,1}} + c_2\mu + \eta\big)\frac{\eta}{\mu},\end{aligned}$$

where we used (4.9) and (5.13) in the last step to obtain the estimate of $\|\nabla\Pi\|_{L^1_t(B^{-1+\frac{3}{p}}_{p,1})}$.

Whence thanks to (5.14), we arrive at

$$\begin{aligned}&\big\|\Delta_j\big(u - e^{\mu t\Delta}u_0\big)\big\|_{L^\infty_t(L^p)} + \mu 2^{2j}\big\|\Delta_j\big(u - e^{\mu t\Delta}u_0\big)\big\|_{L^1_t(L^p)}\\ &\quad\leqslant C d_j 2^{j(1-\frac{3}{p})}\big(2\big\|u_0^3\big\|_{B^{-1+\frac{3}{p}}_{p,1}} + c_2\mu + \eta\big)\frac{\eta}{\mu},\end{aligned}$$

which implies (1.5), and we complete the proof of Theorem 1.2. □

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